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# Using Duality to Solve Generalized Fractional Programming Problems*

A. I. BARROS
*D.E.I.O., Faculdade de Ciências, Universidade de Lisboa, Campo Grande, P 1700 Lisbon, Portugal.*

J. B. G. FRENK
*Econometric Institute, Erasmus University Rotterdam, P. O. Box 1738, 3000 DR Rotterdam, The Netherlands.*

S. SCHAIBLE
*University of California Riverside, Riverside CA 92521, U.S.A.*

and

S. ZHANG
*Econometric Institute, Erasmus University Rotterdam, P.O. Box 1738, 3000 DR Rotterdam, The Netherlands.*

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**Abstract.** In this paper we explore the relations between the standard dual problem of a convex generalized fractional programming problem and the "partial" dual problem proposed by Barros *et al.* (1994). Taking into account the similarities between these dual problems and using basic duality results we propose a new algorithm to directly solve the standard dual of a convex generalized fractional programming problem, and hence the original primal problem, if strong duality holds. This new algorithm works in a similar way as the algorithm proposed in Barros *et al.* (1994) to solve the "partial" dual problem. Although the convergence rates of both algorithms are similar, the new algorithm requires slightly more restrictive assumptions to ensure a superlinear convergence rate. An important characteristic of the new algorithm is that it extends to the nonlinear case the Dinkelbach-type algorithm of Crouzeix *et al.* (1985) applied to the standard dual problem of a generalized linear fractional program. Moreover, the general duality results derived for the nonlinear case, yield an alternative way to derive the standard dual of a generalized linear fractional program. The numerical results, in case of quadratic-linear ratios and linear constraints, show that solving the standard dual via the new algorithm is in most cases more efficient than applying directly the Dinkelbach-type algorithm to the original generalized fractional programming problem. However, the numerical results also indicate that solving the alternative dual (Barros *et al.*, 1994) is in general more efficient than solving the standard dual.

**Key words:** Generalized fractional programming, Dinkelbach-type algorithm, quasiconvexity, duality.

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## 1. Introduction

Fractional programming, i.e. the minimization of a ratio of two functions subject to constraints, has been studied extensively during the last several decades (Avriel *et al.*, 1988; Craven, 1988; Pardalos and Phillips, 1991; Schaible, 1978, 1983; Schaible and Ibaraki, 1983). Lately the focus has shifted towards multi-ratio optimization problems. In particular to generalized fractional programs, where the largest of several ratios of functions is to be minimized (Barros, 1995; Barros *et al.*, 1994; Bernard and Ferland, 1989; Benadada, 1989; Crouzeix and Ferland, 1991; Crouzeix *et al.*, 1983, 1985, 1986). These types of problems arise in economic equilibrium problems, in management applications of goal programming and multi-objective programming involving ratios of functions, and in rational approximation in numerical analysis (Crouzeix *et al.*, 1983).

Among the solution procedures to tackle generalized fractional programs the most popular is the parametric approach. This approach gives rise to a class of algorithms, which are surveyed by Crouzeix and Ferland (1991). Computational experience with some of these algorithms is reported in Benadada (1989); Bernard and Ferland (1989); Ferland and Potvin (1985). An important class of generalized fractional programs is given by convex generalized fractional programs. For this special class a dual description is given by means of standard Lagrangian duality results (Avriel *et al.*, 1988; Craven, 1988; Jagannathan and Schaible, 1983; Werner, 1988). However, these standard duality results did not appear to provide efficient computational tools to solve this class of problems. Quite recently, Barros *et al.* (1994) proposed a different dual description and at the same time used this alternative dual problem to solve the primal problem. In particular the associated dual algorithm can be seen as the dual of a Dinkelbach-type procedure (Crouzeix *et al.*, 1985) and its behavior in case of quadratic-linear ratios and linear constraints appears to be superior to the primal Dinkelbach-type algorithm (Crouzeix *et al.*, 1985) and also to its scaled version (Crouzeix *et al.*, 1986). Based on this approach we introduce in this paper a new dual algorithm solving the standard dual problem, which as the algorithm of Barros *et al.* (1994) detects at the same time an optimal primal solution. It will also be shown that this new algorithm extends to the nonlinear case the Dinkelbach-type algorithm applied to the standard dual problem of a generalized linear fractional program, proposed in Crouzeix *et al.* (1985). At the same time, this enables us to show that the standard dual of a generalized linear fractional programming problem (Crouzeix *et al.* (1983, 1985); Jagannathan and Schaible (1983)) can actually be derived using classical Lagrangian results if the feasible region is a polytope. To summarize, the algorithm discussed in this paper shows how to use the standard dual of a convex generalized fractional programming problem to solve the primal problem. However, it turns out from a computational point of view that this new dual algorithm is inferior to the dual algorithm proposed in Barros *et al.* (1994) to solve the alternative dual.

The paper is organized in the following way. We start by briefly reviewing the dual algorithm proposed in Barros *et al.* (1994) and the Dinkelbach-type algorithm introduced in Crouzeix *et al.* (1985) for generalized fractional programming problems. In Section 3.1 the approach followed in Barros *et al.* (1994) is related to the parametric problem associated with the standard dual problem of a convex generalized fractional problem described in Avriel *et al.* (1988); Craven (1988); Crouzeix *et al.* (1983); Jagannathan and Schaible (1983); Werner (1988). This relation will enable the construction of a new algorithm to solve a convex generalized fractional programming problem by means of its standard dual problem. In Section 3.2 we start by showing how the general duality results can be directly used to derive the standard dual of a generalized linear fractional programming problem (Crouzeix *et al.*, 1983, 1985; Jagannathan and Schaible, 1983). Moreover, it is also shown that in the linear case the new algorithm corresponds to the Dinkelbach-type algorithm applied to the standard dual of a generalized linear fractional program. A scaled version of the new algorithm is also briefly discussed in Section 4. Computational results comparing the performance of the new algorithm with the dual algorithm of Barros *et al.* (1994) and the usual Dinkelbach-type approach are presented in Section 5. To conclude we give some final remarks.

## 2. Generalized Fractional Programming

Let $\mathcal{X} \subset \mathcal{R}^n$ be a compact set and $f_i, g_i \colon \mathcal{K} \to \mathcal{R}$, $i \in I := \{1, \ldots, m\}$, $m \geq 1$, a class of continuous functions where $\mathcal{K}$ is an open set containing $\mathcal{X}$. Assuming $g_i(x) > 0$ for every $x \in \mathcal{X}$ and $i \in I$, consider the generalized fractional program

$$\inf_{x \in \mathcal{X}} \max_{i \in I} \frac{f_i(x)}{g_i(x)}. \tag{P}$$

Since the function $x \mapsto \max_{i \in I} \frac{f_i(x)}{g_i(x)}$ is finite-valued and continuous on the compact set $\mathcal{X} \subseteq \mathcal{R}^n$ the optimization problem (P) has an optimal solution with optimal objective value $\vartheta$(P). We will also assume that the feasible set $\mathcal{X}$ is convex and that either the vector-valued function $f(z)^\top := (f_1(x), \ldots, f_m(x))$ is nonnegative and convex on $\mathcal{X}$, and $g(x)^\top := (g_1(x), \ldots, g_m(x))$ positive concave on $\mathcal{X}$ or that $f$ is convex and $g$ positive affine on $\mathcal{X}$. Observe that, contrary to convex generalized fractional programming it is not required to particularize the feasible set $\mathcal{X}$.

The dual approach introduced in Barros *et al.* (1994) is based on the following equality, which follows from the assumptions and Sion's minimax theorem (Sion, 1958):

$$\min_{x \in \mathcal{X}} \left\{ \max_{i \in I} \frac{f_i(x)}{g_i(x)} \right\} = \max_{y \in \Sigma} \left\{ \min_{x \in \mathcal{X}} \frac{y^\top f(x)}{y^\top g(x)} \right\} \tag{1}$$

with $\Sigma := \{y \in \mathcal{R}^m : y \geq 0, \Sigma_{i \in I} y_i = 1\}$. The above equality relation permits to establish a new dual for the problem (P). In fact, let $c : \Sigma \longrightarrow \mathcal{R}$ be defined by

$$c(y) := \min_{x \in \mathcal{X}} \frac{y^\top f(x)}{y^\top g(x)}. \tag{2}$$

As shown in Lemma 3.1 of Pshenichnyi (1971), the function $c$ is continuous on $\Sigma$ and, moreover, it is semistrictly quasiconcave (Avriel *et al.*, 1988), since it is the infimum of semistrictly quasiconcave functions. Thus, by (1), the optimization problem

$$\max_{y \in \Sigma} c(y) = \max_{y \in \Sigma} \left\{ \min_{x \in \mathcal{X}} \frac{y^\top f(x)}{y^\top g(x)} \right\} \tag{Q}$$

is a semistrictly quasiconcave optimization problem, where a local maximum is a global maximum (Avriel *et al.*, 1988). Moreover, the above optimization problem can be seen as a "partial" dual program of the generalized fractional program (P), since it only "dualizes" the ratios. Observe also that optimization problem (Q) is a particular generalized fractional program involving an infinite number of ratios.

The dual method proposed in Barros *et al.*, (1994) solves (Q) by constructing a sequence $y_k \in \Sigma$, $k \geq 0$ such that the sequence $\{c(y_k) : k \geq 0\}$ is nondecreasing and $\lim_{k \uparrow \infty} c(y_k) = \vartheta(\mathrm{P})$. In order to construct such a sequence, the parametric problem associated with (Q) is considered:

$$\max_{y \in \Sigma} F(y, \lambda) \tag{Q$_\lambda$}$$

where the function $F : \Sigma \times \mathcal{R} \longrightarrow \mathcal{R}$ is given by:

$$F(y, \lambda) := \min_{x \in \mathcal{X}} \left\{ y^\top (f(x) - \lambda g(x)) \right\}. \tag{3}$$

Hence, at the point $y_k \in \Sigma$ with objective value $c(y_k)$ the parametric problem ($\mathrm{Q}_{c(y_k)}$) has to be solved. If the optimal value is zero then the current iteration point $y_k$ is the optimal solution of (Q). Otherwise, the next iteration point is given by $y_{k+1}$ with $y_{k+1}$ an optimal solution of ($\mathrm{Q}_{c(y_k)}$). Moreover, computing the value $c(y_{k+1})$ is equivalent to finding the unique root of $F(y_{k+1}, \lambda) = 0$. Since solving ($\mathrm{Q}_{c(y_k)}$) directly usually takes a lot of time, an indirect approach to solve this problem, making use of the Karush–Kuhn–Tucker conditions, is presented in Barros *et al.* (1994). Besides being efficient, this approach also recovers a primal solution associated with the current iteration point $y_k$, and this enables to exhibit at the end of the procedure an optimal solution of (P). To be more precise, it was shown in Barros *et al.* (1994) that under some reasonable assumptions it is possible to relate an optimal solution $x_{k+1}$ of ($\mathrm{P}_{c(y_k)}$) to an optimal solution $y_{k+1}$ of ($\mathrm{Q}_{c(y_k)}$), with ($\mathrm{P}_\lambda$) denoting the parametric problem of the Dinkelbach-type algorithm (Crouzeix *et al.*, 1985), i.e.

$$F(\lambda) = \inf_{x \in \mathcal{X}} \left\{ \max_{i \in I} \{f_i(x) - \lambda g_i(x)\} \right\}. \tag{P$_\lambda$}$$

Moreover, by the convexity/concavity assumptions on $f$, $g$ and invoking von Neumann's minimax theorem it can be shown that the parametric problems $(\mathrm{Q}_{c(y_k)})$ and $(\mathrm{P}_{c(y_k)})$ provide the same optimal objective value, i.e.

$$F(y_{k+1}, c(y_k)) = F(c(y_k)).$$

The dual algorithm (Barros *et al.*, 1994) is now described by the following procedure:

ALGORITHM 1.

**Step** 0.
Take $y_0 \in \Sigma$, compute $c(y_0) = \min_{x \in \Sigma} \frac{y_0^\top f(x)}{y_0^\top g(x)}$ and let $k := 1$;
**Step** 1.
Determine $y_k :=$ $\mathrm{argmax}_{y \in \Sigma} F(y, c(y_{k-1}))$;
**Step** 2.
If $F(y_k, c(y_{k-1})) = 0$
Then $y_{k-1}$ is an optimal solution of (Q) with value $c(y_{k-1})$ and **Stop**.
Else GoTo **Step** 3;
**Step** 3.
Compute $c(y_k)$, let $k := k + 1$, and GoTo **Step** 1.

The above algorithm converges at least linearly, and sufficient conditions establishing superlinear convergence can also be found in Barros *et al.* (1994).

It is interesting to remark that Algorithm 1 works in a similar way as the Dinkelbach-type algorithm (Crouzeix *et al.*, 1985), which can be summarized as follows:

ALGORITHM 2.

**Step** 0.
Take $x_0 \in \mathcal{X}$, compute $\lambda_1 := \max_{i \in I} \frac{f_i(x_0)}{g_i(x_0)}$ and let $k := 1$;
**Step** 1.
Determine $x_k := \mathrm{argmin}_{x \in \mathcal{X}} \{\max_{i \in I}\{f_i(x) - \lambda_k g_i(x)\}\}$;
**Step** 2.
If $F(\lambda_k) = 0$
Then $x_k$ is an optimal solution of (P) with value $\lambda_k$ and **Stop**.
Else GoTo **Step** 3;
**Step** 3.
Let $\lambda_{k+1} := \max_{i \in I} \frac{(f_i(x_k)}{g_i(x_k)}$, let $k := k + 1$, and GoTo **Step** 1.

Observe that both algorithms proceed in a comparable way. Indeed, at **Step** 1 a parametric problem must be solved to check in **Step** 2 whether or not optimality was reached. If the present iteration point is not optimal, then the next iteration point is given by an optimal solution of the parametric problem solved in **Step** 1. Using this point, a "better" approximation of the optimal objective value is computed in **Step** 3. On the other hand, an essential difference between both algorithms is that the Dinkelbach-type algorithm constructs a nonincreasing sequence $\{\lambda_k : k \geq 1\}$ approaching the optimal objective value $\vartheta$(P) from above, while the dual algorithm constructs a nondecreasing sequence $\{c(y_k) : k \geq 0\}$ approaching $\vartheta$(P) from below.

The scaled versions of the above mentioned algorithms can be found, respectively, in Barros *et al.* (1994) and Crouzeix *et al.* (1986). According to Barros *et al.* (1994) the scaling of the parametric problem appears to be only effective for the Dinkelbach-type algorithm. In fact, the scaled version of the dual algorithm presented in Barros *et al.* (1994) does not appear to produce significant improvements on the behavior of the original dual algorithm. Moreover, according to Barros *et al.* (1994) the original version of the dual algorithm appears to dominate the scaled version of the Dinkelbach-type algorithm, the so-called Dinkelbach-type-2 algorithm, both in terms of iterations and execution time.

## 3. Using Duality to Solve Generalized Fractional Programs

Apart from the recent approach in Barros *et al.* (1994), most of the algorithms that solve generalized fractional programs are "primal" algorithms which do not solve the associated standard dual problem. This may be justified by the fact that the standard dual of a convex generalized fractional program looks much more difficult to handle than its primal counterpart. An exception is given by generalized linear fractional programs for which the dual under certain conditions is again a generalized linear fractional program (Crouzeix *et al.*, 1983, 1985; Jagannathan and Schaible, 1983). This lead Crouzeix *et al.* (1985) to consider solving the dual problem via the Dinkelbach-type algorithm, whenever the unbounded feasible set $\mathcal{X}$ makes it impractical to solve the primal problem directly.

In this section we will show that in spite of the "awkward" form of the standard dual problem of a generalized fractional program we can construct an efficient algorithm to solve the dual problem. This algorithm is based upon the approach proposed by Barros *et al.* (1994) and generates a sequence of iteration points converging from below to the optimal objective value $\vartheta$(P). After introducing this new algorithm for the nonlinear case, we will specialize it to the linear case and show that it corresponds to the Dinkelbach-type algorithm applied to the standard dual of a generalized linear fractional program. Moreover, we will show how the standard dual of a generalized linear fractional program can be directly derived, using the general duality results for the nonlinear case.

### 3.1. Nonlinear Case

In this section we will assume that (P) is a convex generalized fractional programming problem, where the feasible nonempty set $\mathcal{X}$ is given by $\mathcal{X} := \{x \in S :$

$h(x) \leq \mathbf{0}\}$, with $S \subset \mathcal{K}$ a compact convex set and $h : \mathcal{R}^n \longrightarrow \mathcal{R}^r$ a vector-valued convex function. Moreover, the continuous functions $f_i, g_i : \mathcal{K} \longrightarrow \mathcal{R}$, $i \in I$, verify either of the following convexity/concavity assumptions:

(C1) For every $i \in I$, the function $f_i$: $\mathcal{K} \longrightarrow \mathcal{R}$ is convex on S and nonnegative on $\mathcal{X}$ and the function $g_i$: $\mathcal{K} \longrightarrow \mathcal{R}^m$ is positive and concave on S;

(C2) For every $i \in I$, the function $f_i : \mathcal{K} \longrightarrow \mathcal{R}$ is convex on S and the function $g_i : \mathcal{K} \longrightarrow \mathcal{R}^m$ is positive and affine on S.

Clearly, under these conditions the set $\mathcal{X}$ is compact and convex and therefore (P) has an optimal solution with $\vartheta$(P) finite. Observe also that, due to the convexity/concavity assumptions (C1), (C2) we also have that $\vartheta$(P) is nonnegative if $g$ is a positive concave vector-valued function.

In order to simplify the notation, we will introduce $f(x)^\top := (f_1(x), \ldots, f_m(x))$ and $g(x)^\top := (g_1(x), \ldots, g_m(x))$.

An easy direct approach to derive the dual problem of (P) is given by Jagannathan and Schaible (1983). Due to S compact one can apply a generalized Farkas lemma (Bohnenblust *et al.*, 1950) to a system of convex inequalities. This gives rise to the standard dual problem of (P) given by

$$\sup \left\{ t : ty^\top g(x) < y^\top f(x) + z^\top h(x), t \in \mathcal{R}, x \in S, y \in \Sigma, z \geq 0 \right\}$$

with $\Sigma := \{y \in \mathcal{R}^m : y \geq 0, \Sigma_{i \in I} y_i = 1\}$. Since $g_i(x) > 0$ for every $x \in S$ and $i \in I$ the above problem can be rewritten as

$$\sup_{y \in \Sigma, z \geq 0} \left\{ \inf_{x \in S} \frac{y^\top f(x) + z^\top h(x)}{y^\top g(x)} \right\}. \tag{D}$$

Moreover, due to S compact and $y^\top g(x) > 0$ for every $x \in S$, one may replace inf by min. Observe that, although $\vartheta$(D) equals $\vartheta$(P) (Crouzeix *et al.*, 1983; Jagannathan and Schaible, 1983) there might not exist an optimal dual solution.

The similarities in structure between the standard dual problem (D) and the "partial" dual problem introduced in Barros *et al.* (1994) suggest that it is possible to directly solve (D). In this case, we need to guarantee that (D) has an optimal solution. Hence, we will impose a Slater-type condition on the set $\mathcal{X}$.

**Slater condition**

*Let $J$ denote the set of indices $1 \leq j \leq r$ such that the jth component $h_j$ of the vector-valued convex function $h : \mathcal{R}^n \rightarrow \mathcal{R}^r$ is affine. There exists some $x$ belonging to the relative interior* ri$(S)$ *of S satisfying $h_j(x) < 0, j \notin J$ and $h_j(x) \leq 0, j \in J$.*

Using now the indirect Lagrangian approach in Craven (1988), the following result is easy to show.

PROPOSITION 1. *If the Slater condition holds then the parametric problem*

$$\sup_{y\in\Sigma,z\geq 0}\left\{\min_{x\in S}\{y^\top f(x)+z^\top h(x)-\lambda y^\top g(x)\}\right\} \tag{D$_\lambda$}$$

*has an optimal solution and* $\vartheta(D_\lambda)=\vartheta(P_\lambda)$ *for any* $\lambda$ *if* $g$ *is a positive affine vector-valued function on S or for* $\lambda\geq 0$ *if* $g$ *is a positive concave vector-valued function on S. Moreover, the dual problem of (P) has an optimal solution and* $\vartheta(D)=\vartheta(P)$.

*Proof.* By Theorem 28.2 of Rockafellar (1970) the Lagrangian dual ($D_\lambda$) of the parametric problem ($P_\lambda$), given by

$$\sup_{y\in\Sigma,z\geq 0}\left\{\min_{x\in S}\left\{y^\top f(x)+z^\top h(x)-\lambda y^\top g(x)\right\}\right\}$$

has under Slater's condition an optimal solution for every $\lambda\in\mathcal{R}$ if the functions $g_i$, $i\in I$, are positive and affine on S or for every $\lambda\geq 0$ if the functions $g_i$, $i\in I$, are positive and concave on S. It also follows that $\vartheta(D_\lambda)=\vartheta(P_\lambda)$ and this proves the first result.

Using the above result and Theorem 4.1 of Crouzeix *et al.* (1985) we have for $\lambda_*=\vartheta(P)$ that

$$0=\vartheta(P_{\lambda_*})=\vartheta(D_{\lambda_*})=\max_{y\in\Sigma,z\geq 0}\left\{\min_{x\in S}\left\{y^\top f(x)+z^\top h(x)-\lambda_* y^\top g(x)\right\}\right\}$$

Hence, there exists some $y_*\in\Sigma$ and $z_*\geq 0$ such that

$$\min_{x\in S}\left\{y_*^\top f(x)+z_*^\top h(x)-\lambda_* y_*^\top g(x)\right\}=0$$

and

$$\min_{x\in S}\left\{y^\top f(x)+z^\top h(x)-\lambda_* y^\top g(x)\right\}\leq 0 \tag{5}$$

for every $y\in\Sigma$ and $z\geq 0$. By the above equality and $g(x)>0$ for every $x\in S$ we obtain that

$$\lambda_*=\min_{x\in S}\left\{\frac{y_*^\top f(x)+z_*^\top h(x)}{y_*^\top g(x)}\right\}$$

On the other hand, from (5) it follows that

$$\lambda_*\geq\min_{x\in S}\left\{\frac{y^\top f(x)+z^\top h(x)}{y^\top g(x)}\right\}$$

for every $y\in\Sigma$ and $z\geq 0$. Hence, by the previous equality and inequality the dual problem (D) has an optimal solution and $\lambda_*=\vartheta(P)$ equals $\vartheta(D)$.

Since we are interested in an algorithm to solve the standard dual problem (D) we will assume from now on that the Slater condition holds. Hence we can rewrite the standard dual (D) as

$$\max_{y\in\Sigma, z\geq 0} d(y,z)$$

where the function d: $\Sigma \times \mathcal{R}_+^r \longrightarrow \mathcal{R}$ is given by

$$d(y,z) := \min_{x\in S} \frac{y^\top f(x) + z^\top h(x)}{y^\top g(x)} \tag{6}$$

with $\mathcal{R}_+^r$ denoting the nonnegative orthant of $\mathcal{R}^r$. Notice that (6) corresponds to a single-ratio fractional programming problem. Clearly, by the positivity of $g$ on $S$ and $S$ compact the function $d$ is continuous on $\Sigma \times \mathcal{R}_+^r$, see Lemma 3.1 of Pshenichnyi (1971). Moreover, the function $d$ is semistrictly quasiconcave since it is the infimum of semistricly quasiconcave functions $x \mapsto \frac{y^\top f(x)+z^\top h(x)}{y^\top g(x)}$, see Avriel *et al.* (1988). Hence, (D) corresponds to a quasiconcave optimization problem, where a local maximum is a global maximum, see Avriel *et al.* (1988). Notice, since the Slater condition holds we know that this maximum is attained, i.e. there exists some $(y,z) \in \Sigma \times \mathcal{R}_+^r$ such that $d(y,z) = \vartheta(D)$.

Due to the similarities between the standard dual (D) and the "partial" dual (Q) we will use the approach described in Barros *et al.* (1994) to derive a new algorithm to solve (D). Therefore, we will start by relating the parametric problems associated with (D) and with (Q) and hence, we introduce the value function $G : \mathcal{R} \longrightarrow \mathcal{R}$ associated with ($D_\lambda$) given by

$$G(\lambda) := \max_{y\in\Sigma, z\geq 0} G(y,z,\lambda)$$

with

$$G(y,z,\lambda) := \min_{x\in S} \left\{ y^\top f(x) + z^\top h(x) - \lambda y^\top g(x) \right\}.$$

LEMMA 2. *For $\lambda \in \mathcal{R}$ if $g$ is a positive affine vector-valued function on $S$ or for $\lambda \geq 0$ if $g$ is a positive concave vector-valued function on $S$*

$$G(\lambda) = \max_{y\in\Sigma} F(y,\lambda). \tag{7}$$

*Moreover, $\hat{y}$ is an optimal solution of ($Q_\lambda$) if and only if there exists some $\hat{z} \geq 0$ such that $(\hat{y},\hat{z})$ is an optimal solution of ($D_\lambda$).*

*Proof.* Remember by definition we have that

$$G(\lambda) := \max_{y\in\Sigma} \max_{z\geq 0} \left\{ \min_{x\in S} \left\{ y^\top f(x) + z^\top h(x) - \lambda y^\top g(x) \right\} \right\}.$$

Considering $z \geq 0$ as the vector of Lagrangian multipliers associated with the constraints $h(x) \leq 0$, we obtain under the Slater condition by Theorem 28.2 of Rockafellar (1970) that

$$\begin{aligned} F(y, \lambda) &= \min_{x \in \mathcal{X}} \left\{ y^\top f(x) - \lambda y^\top g(x) \right\} \\ &= \max_{z \geq 0} \min_{x \in S} \left\{ y^\top (f(x) - \lambda g(x)) + z^\top h(x) \right\} \end{aligned} \tag{8}$$

and hence it follows that $G(\lambda) = \max_{y \in \Sigma} F(y, \lambda)$.

To verify the second part, notice that, if $\hat{y} \in \Sigma$ is an optimal solution of $(\mathrm{Q}_\lambda)$, then by (7) and (8) there exists some $\hat{z} \geq 0$ satisfying

$$\begin{aligned} \max_{y \in \Sigma, z \geq 0} G(y, z, \lambda) &= F(\hat{y}, \lambda) = \max_{z \geq 0} \left\{ \min_{x \in S} \left\{ \hat{y}^\top (f(x) - \lambda g(x)) + z^\top h(x) \right\} \right\} \\ &= G(\hat{y}, \hat{z}, \lambda) \end{aligned}$$

and so $(\hat{y}, \hat{z})$ is an optimal solution of $(\mathrm{D}_\lambda)$. Moreover, if $(\hat{y}, \hat{z})$ is an optimal solution of $(\mathrm{D}_\lambda)$, then it follows by weak duality and (7) that

$$\max_{y \in \Sigma, z \geq 0} G(y, \lambda) = G(\hat{y}, \hat{z}, \lambda) = \min_{x \in S} \left\{ \hat{y}^\top f(x) + \hat{z}^\top h(x) - \lambda \hat{y}^\top g(x) \right\} \leq F(\hat{y}, \lambda)$$

and this shows that $\hat{y}$ is an optimal solution of $(\mathrm{Q}_\lambda)$.

Notice from the above lemma that $\hat{z}$ is the optimal Lagrangian multiplier vector associated with the constraints $h(x) \leq 0$ of the optimization problem defined by $F(\hat{y}, \lambda)$.

In spite of the different formats of the duals (Q) and (D) they have by Lemma 2 equivalent associated parametric problems, if the Slater condition and the convexity/concavity assumptions, $(\mathrm{C}_1)$, $(\mathrm{C}_2)$, hold. Moreover, since it was shown in Barros *et al.* (1994) that the parametric problems associated with (P) and (Q) are equivalent, using the above lemma, this relation can now be extended to the parametric problem associated with (D). Hence, this important relation also implies that for $(y_k, z_k)$ an optimal solution of $(\mathrm{D}_{\lambda_k})$ then the function $G_{(y_k, z_k)} : \mathcal{R} \longrightarrow \mathcal{R}$ given by

$$G_{(y_k, z_k)}(\lambda) = G(y_k, z_k, \lambda) \tag{9}$$

approximates the "primal" parametric problem function $F$ at $\lambda_k$ from below and $G_{(y_k, z_k)}(\lambda_k) = F(\lambda_k)$. Hence, the root of the equation $G_{(y_k, z_k)}(\lambda) = 0$ given by $d(y_k, z_k)$ yields a lower bound on $\vartheta(\mathrm{D})$. Before introducing the algorithm to solve (D) we will briefly discuss how to compute $d(y_k, z_k)$. Computing $d(y_k, z_k)$ corresponds to solving a single-ratio fractional programming problem, which can be easily done using the classical Dinkelbach algorithm, see Dinkelbach (1967). However, the efficiency of this procedure depends mostly on whether the associated parametric problem has a "nice" form. Clearly, if assumption $(\mathrm{C}_2)$ holds then the associated parametric problem corresponds to a convex problem. On the other hand, if assumption ($C_1$) holds the parametric problem is convex only if the parameter $\lambda$ is nonnegative. Nevertheless, in this case, $\vartheta(P) \geq 0$ and hence we are only interested in $(y, z) \in \Sigma \times \mathcal{R}^r_+$ such that $d(y, z) \geq 0$. Observe that, in this case (6) is also a convex single-ratio fractional programming problem.

In the same way as for the dual algorithm (Barros *et al.*, 1994) it is easy to show that the sequence $\{d(y_k, z_k)\}_{k\geq 1}$ is strictly increasing. In fact, Lemma 3.1 of Barros *et al.* (1994) can be extended to this case as follows.

LEMMA 3. *For* $(\hat{y}, \hat{z}) \in \Sigma \times \mathcal{R}^r_+$ *we have*

$$U_d(\hat{y}, \hat{z})) = \{(y, z) \in \Sigma \times \mathcal{R}^r_+ : G(y, z, d(\hat{y}, \hat{z})) > 0\}$$

and

$$U_d(\hat{y}, \hat{z})) = \{(y, z) \in \Sigma \times \mathcal{R}^r_+ : G(y, z, d(\hat{y}, \hat{z})) \geq 0\}$$

*where* $U_d(d(\hat{y}, \hat{z}))$ *and* $U_d^\circ(d(y, z))$ *denote the upper respectively the strict upper level set of the function* $d$ *at* $(\hat{y}, \hat{z})$.

The proof of the above lemma can be found in Barros (1995).

We can now propose the "dual-type" algorithm to solve (D):

ALGORITHM 3.

**Step** 0.
If assumption ($C_1$) holds
  Then Let $\lambda_0 := 0$
  Else Take $y_0 \in \Sigma$, $z_0 \geq 0$;
    Compute $\lambda_0 := d(y_0, z_0) = \min_{x\in S} \frac{y_0^\top f(x)+z_0^\top h(x)}{y_0^\top g(x)}$;
    Let $k := 1$;

**Step** 1.
Determine $(y_k, z_k) := \operatorname{argmax}_{y\in\Sigma, z\geq 0} G(y, z, \lambda_{k-1})$;

**Step** 2.
If $G(\lambda_{k-1}) = 0$
  Then $(y_{k-1}, z_{k-1})$ is an optimal solution of (D) with value $\lambda_{k-1}$ and Stop.
  Else GoTo **Step** 3;

**Step** 3.
Compute $\lambda_k := d(y_k, z_k)$, let $k := k + 1$, and GoTo **Step** 1.

Although this new dual algorithm is similar to the one presented in Barros *et al.* (1994) the derivation of the convergence results is more "complex". This is mainly due to the fact that the feasible set of the standard dual problem (D) given by $\Sigma \times \mathcal{R}^r_+$ is no longer compact as in the case of (Q).

To prove the convergence of this new dual algorithm we need to investigate the behavior of the approximation function $G_{(y_k, z_k)} : \mathcal{R} \longrightarrow \mathcal{R}$. In a similar way as in Barros *et al.* (1994) it can be shown that this function is a concave lower approximation of the function F. Observe that by Theorem 23.4 of Rockafellar (1970) the subgradient set $\partial(-G_{(y,z)})(\lambda)$ of the convex function $-G_{(y,z)} : \mathcal{R} \longrightarrow \mathcal{R}$ at the point $\lambda$ is nonempty. Remember that $\rho \in \mathcal{R}$ is a subgradient of the function $-G_{(y,z)}$ at the point $\lambda$ if and only if

$$G_{(y,z)}(\lambda + t) \leq G_{(y,z)}(\lambda) - t\rho \tag{10}$$

for every $t \in \mathcal{R}$. The next result characterizes the subgradient set $\partial(-G_{(y,z)})(\lambda)$. Before mentioning this result we introduce for fixed $(y, z) \in \Sigma \times \mathcal{R}_+^r$ the set $S_{(y,z)}$ $(\lambda)$ of optimal solutions of the optimization problem

$$\min_{x \in S} \left\{ y^\top f(x) + z^\top h(x) - \lambda y^\top g(x) \right\}.$$

i.e.

$$S_{(y,z)}(\lambda) := \left\{ x \in S : y^\top (f(x) - \lambda g(x)) + z^\top h(x) = G_{(y,z)}(\lambda). \right\} \tag{11}$$

Clearly, this set is nonempty. Also, by the continuity of the vector-valued functions $f, g$ and $h$ it must be closed, and thus by the compactness of $S$ and $S_{(y,z)})$ $(\lambda) \subset S$ it is compact. Finally, if $\lambda \geq 0$ then the function $x \mapsto y^\top$ $(f(x) - \lambda g(x)) + z^\top h(x)$ is convex due to the convexity and concavity of $f$, $h$ and $g$ respectively, and this implies that $S_{(y,z)}(\lambda) \subset S$ is also convex for every $\lambda \geq 0$. Observe that the above result also holds for any $\lambda$, if the functions $g_i$ are positive and affine on $S$ for every $i \in I$.

LEMMA 4. *For every fixed* $(y, z) \in \Sigma \times \mathcal{R}_+^r$ *and* $\lambda \in \mathcal{R}$ *follows that*

$$\partial(-G_{(y,z)})(\lambda) = \left[ \min_{x \in S_{(y,z)}(\lambda)} \left\{ y^\top g(x) \right\}, \max_{x \in S_{(y,z)}(\lambda)} \left\{ y^\top g(x) \right\} \right].$$

The proof of the above lemma is omitted since this result is a special case of a more general result given by Theorem 7.2 of Rockafellar (1983) or Theorem 4.4.2 of Hiriart-Urruty and Lemaréchal (1993). However, an easy proof of this special case can be found in Barros *et al.* (1994).

Observe, since $S$ is a compact set and $g$ is a positive and continuous vector-valued function that

$$\delta := \min_{x \in S} \min_{i \in I} g_i(x) > 0 \text{ and } \Delta := \max_{x \in S} \max_{i \in I} g_i(x) < +\infty. \tag{12}$$

This implies using $y \in \Sigma$ that

$$\partial(-G_{(y,z)}(\lambda) \subseteq [\delta, \Delta]$$

for every $(y, z) \in \Sigma \times \mathcal{R}_+^r$ and $\lambda$ appropriately chosen.

Denote now by $k^*$ the number of times that **Step** 1 was executed by the algorithm. Clearly if $k^*$ is finite it follows that $G(y_{k^*}, z_{k^*}, d(y_{k^*-1}, z_{k^*-1})) = 0$, while for $k^* = +\infty$ the algorithm does not stop. Before mentioning the next result we introduce

$$\Delta_k(y,z) := \max\left\{y^\top g(x) : x \in S_{(y,z)}(d(y_k, z_k))\right\}$$

and

$$\begin{aligned}\delta_{k+1} : &= \min\left\{y_{k+1}^\top g(x) : x := \operatorname{argmin}_{x\in S}\frac{y_{k+1}^\top f(x) + z_{k+1}^\top h(x)}{y_{k+1}^\top g(x)}\right\} \\ &= \min\left\{y_{k+1}^\top g(x) : x \in S_{(y_{k+1},z_{k+1})}(d(y_{k+1}, z_{k+1}))\right\}\end{aligned}$$

Observe that by Lemma 4 we have

$$\begin{aligned}&\Delta_k(y,z) \in \partial(-G_{(y,z)})(d(y_k,z_k)) \text{ and} \\ &\delta_{k+1} \in \partial(-G_{(y_{k+1},z_{k+1})})(d(y_{k+1},z_{k+1})).\end{aligned}$$

THEOREM 5. *The sequence* $(y_k, z_k), 0 \le k < k^*$, *does not contain optimal solutions of (D) and the corresponding function values* $d(y_k, z_k), 0 \le k < k^*$ *are strictly increasing. Moreover, if* $k^*$ *is finite, then* $d(y_{k^*}, z_{k^*}) = \vartheta(P)$ *while for* $k^* = +\infty$ *we have*

$$\lim_{k\uparrow\infty} d(y_k, z_k) = \vartheta(P).$$

*Finally, if* $k^* = +\infty$ *and* $(y_*, z_*)$ *is an optimal solution of (D), then*

$$\begin{aligned}\vartheta(P) - d(y_{k+1}, z_{k+1}) &\le \left(1 - \frac{\Delta_k(y_*, z_*)}{\delta_{k+1}}\right)(\vartheta(P) - d(y_k, z_k)) \\ &\le \left(1 - \frac{\delta}{\Delta}\right)(\vartheta(P) - d(y_k, z_k))\end{aligned} \tag{13}$$

*holds for every* $k \ge 0$, *with* $\delta$ *and* $\Delta$ *defined by (12).*

*Proof.* Using Lemma 3 it follows that $G(y_{k+1}, z_{k+1}, d(y_k, z_k)) > 0$ if and only if $(y_k, z_k)$ is nonoptimal. Moreover, by the same lemma we obtain that

$$d(y_{k+1}, z_{k+1}) > d(y_k, z_k)$$

if $(y_k, z_k)$ is nonoptimal, and so the first part of the theorem is proved.

Consider now the case with $k^*$ finite. Since the algorithm stopped in a finite number of steps we must have $G(y_{k^*}, z_{k^*}, d(y_{k^*-1})) = 0$ and using again Lemma 3 it follows that $(y_{k^*}, z_{k^*})$ solves (D). Hence, by Proposition 1 we have $d(y_{k^*}, z_{k^*}) = \vartheta(D) = \vartheta(P)$.

To verify the last part of the result, notice that $d(y_k, z_k), k \ge 0$, is strictly increasing for $k^* = +\infty$, and since $d(y_k, z_k) \le \vartheta(D) < \infty$ for every $k \ge 0$, it must follow that $\lim_{k\uparrow\infty} d(y_k, z_k)$ exists and is finite-valued. Moreover, by Lemma 4 and (10) we obtain for every optimal solution $(y_*, z_*)$ of $(D)$ that

$$\begin{aligned}&G_{(y_*,z_*)}(d(y_*, z_*)) - G_{(y_*,z_*)}(d(y_k, z_k)) \\ &\quad \le -(d(y_*, z_*) - d(y_k, z_k))\Delta_k(y_*, z_*).\end{aligned}$$

Since $G_{(y_*,z_*)}(d(y_*, z_*)) = 0$ this implies that

$$
\begin{aligned}
G_{(y_{k+1},z_{k+1})}(d(y_k, z_k)) &= \max_{y\in\Sigma, z\geq 0} G_{(y,z)}(d(y_k, z_k)) \geq G_{(y_*,z_*)}(d(y_k, z_k)) \\
&\geq (d(y_*, z_*) - d(y_k, z_k))\Delta_k(y_*, z_*). \qquad (14)
\end{aligned}
$$

On the other hand, applying again Lemma 4 and (10) we obtain

$$
\begin{aligned}
G_{(y_{k+1},z_{k+1})}(d(y_k, z_k)) &= G_{(y_{k+1},z_{k+1})}(d(y_k, z_k)) \\
&\quad -G_{(y_{k+1},z_{k+1})}(d(y_{k+1}, z_{k+1})) \\
&\leq (d(y_{k+1}, z_{k+1}) - d(y_k, z_k))\delta_{k+1}.
\end{aligned}
$$

The above inequality and (14) imply that

$$
(d(y_{k+1}, z_{k+1}) - d(y_k, z_k))\delta_{k+1} \geq (d(y_*, z_*) - d(y_k, z_k))\Delta_k(y_*, z_*). \qquad (15)
$$

Since $\Delta_k(y_*, z_*)$ and $\delta_{k+1}$ belong to the interval $[\delta, \Delta]$ it follows by (15) and the existence of $\lim_{k\uparrow\infty} d(y_k, z_k)$ that

$$
\lim_{k\uparrow\infty} d(y_k, z_k) = d(y_*, z_*) = \vartheta(D) = \vartheta(P)
$$

Moreover, by the same argument we obtain

$$
\begin{aligned}
\vartheta(P) - d(y_{k+1}, z_{k+1}) &= \vartheta(P) - d(y_k, z_k) + d(y_k, z_k) - d(y_{k+1}, z_{k+1}) \\
&\leq \left(1 - \frac{\Delta_k(y_*, z_*)}{\delta_{k+1}}\right)(\vartheta(P) - d(y_k, z_k)) \\
&\leq \left(1 - \frac{\delta}{\Delta}\right)(\vartheta(P) - d(y_k, z_k)).
\end{aligned}
$$

Clearly, by inequality (13) this algorithm converges at least linearly. In order to improve this convergence rate we need to impose a "stronger" constraint qualification than the Slater condition. Therefore, we will consider the following so-called strong Slater condition (Hiriart-Urruty and Lemaréchal, 1993):

**Strong Slater condition**
*There exists some* $\hat{x} \in \mathrm{ri}(S)$ *satisfying* $h(\hat{x}) < 0$.

This condition is by the continuity of the vector-valued function $h$ equivalent to the requirement that there exists some $\hat{x} \in S$ satisfying $h(\hat{x}) < 0$. We can now establish the following simple condition which improves the convergence rate. However, before mentioning this result we will show that the strong Slater condition implies the existence of an accumulation point of the sequence $\{(y_k, z_k)\}_{k\geq 0}$ generated by Algorithm 3. Observe that the existence of such an accumulation point is not immediately clear due to $z_k \geq 0$ for every $k \geq 0$.

LEMMA 6. *If the strong Slater condition holds, i.e. there exists some $\hat{x} \in S$ satisfying $h(\hat{x}) < 0$, then the sequence $\{(y_k, z_k)\}_{k \geq 0}$ has an accumulation point $(y_*, z_*)$ and this accumulation point is an optimal solution of (D). Moreover, if (D) has a unique optimal solution $(y_*, z_*)$ then $\lim_{k\uparrow\infty} y_k = y_*$ and $\lim_{k\uparrow\infty} z_k = z_*$.*

*Proof.* Suppose that the sequence $(y_k, z_k) \in \Sigma \times \mathcal{R}_+^r$ has no convergent subsequence in $\Sigma \times \mathcal{R}_+^r$, $k \in K$. This implies, since $\Sigma$ is a compact set, that there exists a subsequence $K_1 \subseteq K$ with

$$\lim_{k \in K_1, k\uparrow\infty} y_k = y_* \text{ and } \lim_{k \in K_1, k\uparrow\infty} \|z_k\| = \infty \tag{16}$$

However,

$$d(y_k, z_k) = \min_{x \in S} \frac{y_k^\top f(x) + z_k^\top h(x)}{y_k^\top g(x)} \leq \frac{y_k^\top f(\hat{x}) + z_k^\top h(\hat{x})}{y_k^\top g(\hat{x})}$$

and due to $g(\hat{x}) > 0$, $h(\hat{x}) < 0$ and (16) it follows that

$$\lim_{k \in K_1, k\uparrow\infty} d(y_k, z_k) = -\infty.$$

However, the sequence $d(y_k, z_k)$ is strictly increasing by Theorem 5 and this yields a contradiction. Hence the sequence $\{(y_k, z_k)\}_{k\geq 0}$ has an accumulation point $(y_*, z_*) \in \Sigma \times \mathcal{R}_+^r$. Applying again Theorem 5 and using the continuity of the function $d$ we obtain that such an accumulation point $(y_*, z_*)$ is an optimal solution of $(D)$, which concludes the proof of the first part of this lemma.

The second part of this lemma is easily verified by contradiction.

Using the previous lemma we can establish the following superlinear convergence rate result.

PROPOSITION 7. *If the strong Slater condition holds and for every optimal solution $(y_*, z_*)$ of $(D)$ the optimization problem*

$$\min_{x \in S} \frac{y_*^\top f(x) + z_*^\top h(x)}{y_*^\top g(x)} \tag{D$_*$}$$

*has a unique optimal solution then the new dual algorithm converges superlinearly.*

*Proof.* If for all optimal solutions $(y_*, z_*)$ of $(D)$

$$\lim_{k\uparrow\infty} \sup \left(1 - \frac{\Delta_k(y_*, z_*)}{\delta_{k+1}}\right) = 0$$

then, from Theorem 5, it follows that the convergence rate of the new dual algorithm is superlinear, and so the result is proved. Let $\delta_\infty := \limsup_{k\uparrow\infty} \delta_{k+1}$. By the definition of lim sup there exists a subsequence $K \subseteq \mathcal{N}$ such that $\delta_\infty = \lim_{k\in K, k\uparrow\infty} \delta_{k+1}$. Moreover, by Lemma 6 it follows that the sequence $\{(y_k, z_k)\}_{k\geq 0}$ admits a subsequence $K_1 \subseteq K$ such that $\lim_{k \in K_1, k\uparrow\infty}(y_{k+1}, z_{k+1}) = (y_*, z_*)$ with $(y_*, z_*)$ an accumulation point, and this accumulation point is an optimal solution of $(D)$. Hence, consider the sequence $1-\frac{\Delta_k(y_*,z_*)}{\delta_{k+1}}$ for such an accumulation point $(y_*, z_*)$. It is easy to verify that the point-to-set mapping $(y, z) \mapsto \partial(-G_{(y,z)})(d(y, z))$ is upper semicontinuous. Since

$$\delta_{k+1} \in \partial(-G_{(y_{k+1},z_{k+1})})(d(y_{k+1}, z_{k+1})), \lim_{k\in K_1, k\uparrow\infty} \delta_{k+1} = \delta_\infty \text{ and}$$

$$\lim_{k\in K_1, k\uparrow\infty} (y_{k+1}, z_{k+1}) = (y_*, z_*)$$

we obtain that

$$\delta_\infty \in \partial(-G((_{(y_*,z_*)})(d(y_k, z_k)). \tag{17}$$

On the other hand, it is clear by Lemma 4 that

$$\Delta_k(y_*, z_*) \in \partial(-G_{y_*,z_*)})(d(y_k, z_k)).$$

Moreover, since the increasing sequence $d(y_k, z_k)$ converges from below to $d(y_*, z_*)$, it follows by the convexity of the function $-G_{(y_*,z_*)}$ and

$$\Delta_k(y_*, z_*) \in \partial(-G_{(y_*,z_*)})(d(y_k, z_k))$$

that

$$\Delta_k(y_*, z_*) \leq \Delta_{k+1}(y_*, z_*) \leq \ldots \leq a_* \text{ with } a_* \in \partial(-G_{(y_*,z_*)})(d(y_*, z_*)).$$

This implies $\lim_{k\uparrow\infty} \Delta_k(y_*, z_*) =: \Delta_\infty(y_*, z_*)$ exists and by the upper semicontinuity of the point-to-set mapping $\lambda \mapsto \partial(-G_{(y_*,z_*)})(\lambda)$ we obtain that

$$\Delta_\infty(y_*, z_*) \in \partial(-G_{(y_*,z_*)})(d(y_*, z_*)).$$

Since we already observed that

$$\Delta_\infty(y_*, z_*) \leq a_* \text{ for every } a_* \in \partial(-G_{(y_*,z_*)})(d(y_*, z_*)),$$

it must follow by Lemma 4 that

$$\Delta \geq \Delta_\infty(y_*, z_*) = \min\left\{y_*^\top g(x) : x \in S_{(y_*,z_*)}(d(y_*, z_*))\right\} \geq \delta > 0. \tag{18}$$

Observe now that by (17) and (18) we have

$$\begin{aligned} 0 \leq \limsup_{k\uparrow\infty}\left(1-\frac{\Delta_k(y_*, z_*)}{\delta_{k+1}}\right) &= 1-\liminf_{k\uparrow\infty}\frac{\Delta_k(y_*, z_*)}{\delta_{k+1}} \\ &= 1-\frac{\Delta_\infty(y_\infty, z_*)}{\delta_\infty} < 1, \end{aligned}$$

and hence, if the optimization problem $(D_*)$ has a unique optimal solution, implying by (17), (18) and Lemma 4 that $0 < \delta \leq \Delta_\infty(y_*, z_*) = \delta_\infty \leq \Delta < \infty$ the desired result follows.

In order to guarantee the uniqueness condition expressed in the above proposition we will consider the subclass of quasiconvex functions usually known as strictly quasiconvex, see Avriel *et al.* (1988). Observe by Proposition 3.29 of Avriel *et al.* (1988) that $\min_{x \in S} \psi(x)$ has a *unique* optimal solution if $\psi : \mathcal{K} \longrightarrow \mathcal{R}$ is strictly quasiconvex and continuous. The next corollary establishes sufficient conditions on the functions $f_i$ and $g_i$ to ensure that the convergence rate of Algorithm 3 is superlinear.

COROLLARY 8. *If the strong Slater condition and either one of the following conditions*

*(1)* $f : \mathcal{K} \longrightarrow \mathcal{R}^m$ *is strictly convex on S and nonnegative on* $\mathcal{X}$ *and* $g : \mathcal{K} \longrightarrow \mathcal{R}^m$ *is positive and concave on S;*
*(2)* $f : \mathcal{K} \longrightarrow \mathcal{R}^m$ *is convex on S and nonnegative on* $\mathcal{X}$ *and* $g : \mathcal{K} \longrightarrow \mathcal{R}^m$ *is positive and strictly concave on S;*
*(3)* $f : \mathcal{K} \longrightarrow \mathcal{R}^m$ *is strictly convex on S and* $g : \mathcal{K} \longrightarrow \mathcal{R}^m$ *is positive and affine on S*

*hold then the new dual algorithm converges superlinearly.*

*Proof.* We will prove the result only for (1) and (2). For (3) the result follows easily. By Proposition 7 and the properties of strictly quasiconvex functions it is sufficient to show that for any optimal solution $(y_*, z_*)$ of $(D)$ the function $\psi : \mathcal{K} \longrightarrow \mathcal{R}$ given by

$$\psi(x) := \frac{y_*^\top f(x) + z_*^\top h(x)}{y_*^\top g(x)}$$

is strictly quasiconvex on $S$. By Proposition 1 we have that

$$\min_{x \in S} \frac{y_*^\top f(x) + z_*^\top h(x)}{y_*^\top g(x)}$$

equals $\vartheta(P) \geq 0$, and so it follows that $x \mapsto y_*^\top f(x) + z_*^\top h(x)$ is nonnegative on $S$. This yields for every $x_1, x_2 \in S$ with $x_1 \neq x_2$ and $0 < \lambda < 1$ that

$$\begin{aligned}
&\psi(\lambda x_1 + (1-\lambda)x_2) \\
&< \frac{\lambda y_*^\top f(x_1) + \lambda z_*^\top h(x_1) + (1-\lambda) y_*^\top f(x_2) + (1-\lambda) z_*^\top h(x_2)}{\lambda y_*^\top g(x_1) + (1-\lambda) y_*^\top g(x_2)} \\
&= \frac{\lambda y_*^\top g(x_1) \frac{y_*^\top f(x_1) + z_*^\top h(x_1)}{y_*^\top g(x_1)} + (1-\lambda) y_*^\top g(x_2) \frac{y_*^\top f(x_2) + z_*^\top h(x_2)}{y_*^\top g(x_2)}}{\lambda y_*^\top g(x_1) + (1-\lambda) y_*^\top g(x_2)} \\
&\leq \max\left\{ \frac{y_*^\top f(x_1) + z_*^\top h(x_1)}{y_*^\top g(x_1)}, \frac{y_*^\top (x_2) + z_*^\top h(x_2)}{y_*^\top g(x_2)} \right\} \\
&= \max\{\psi(x_1), \psi(x_2)\}
\end{aligned}$$

which completes the proof.

The above conditions to ensure a superlinear rate of convergence resemble the ones needed to establish the same result for the dual algorithm in Barros *et al.* (1994). However, for the new algorithm the strong Slater condition is additionally required. On the other hand, the new algorithm proposed may be more suitable whenever the feasible set $\mathcal{X}$ is formed by "easy" and "difficult" constraints. Clearly, in this case by grouping in $S$ the "easy" constraints yields an easier single-ratio fractional programming problem. However, the practicality of this algorithm depends mostly on how **Step** 1 is solved. Since $(D_\lambda)$ and $(Q_\lambda)$ are equivalent problems, we will use the same type of approach developed in Barros *et al.* (1994) to solve $(Q_\lambda)$.

As observed $(D_\lambda)$ corresponds to a Lagrangian dual of $(P_\lambda)$ and thus we can relate an optimal solution of $x_{k+1}$ of $(P_{\lambda_k})$ to an optimal solution $(y_{k+1}, z_{k+1})$ of $(D_{\lambda_k})$. To derive this relation, we will assume additionally that $f_i$, $g_i$, $i = 1, \dots, m$ and $h_j : \mathcal{R}^n \longrightarrow \mathcal{R}_i, j = 1, \dots, s$, are also differentiable functions and that the nonempty compact convex set $S$ is given by

$$S := \{x \in \mathcal{R}^n : p_l(x) \leq 0, l = 1, \dots, s\}$$

where $p_l : \mathcal{R}^n \longrightarrow \mathcal{R}$, $l = 1, \dots, s$, are convex and differentiable functions.

Clearly $(P_{\lambda_k})$ is equivalent to the following convex programming problem

$$\begin{aligned} & \min t \\ s.t. : q_i(x) - t &\leq 0 \quad \forall i = 1, \dots, m \\ h_j(x) &\leq 0 \quad \forall j = 1, \dots, r \\ p_l(x) &\leq 0 \quad \forall l = 1, \dots, s \end{aligned}$$

with $q_i(x) := f_i(x) - \lambda_k g_i(x), i = 1, \dots, m$. Let $x_{k+1}$ and $t_{k+1}$ be an optimal solution of the above problem, and define $I' := \{1 \leq i \leq m : q_i(x_{k+1}) = t_{k+1}\}$, $J' := \{1 \leq j \leq r : h_j(x_{k+1}) = 0\}$ and $L' := \{1 \leq l \leq s : p_l(x_{k+1}) = 0\}$. Since the Slater condition holds the Karush–Kuhn–Tucker conditions ensure the existence of nonnegative scalars $u_i; i \in I', v_j; j \in J'$ and $\xi_l; l \in L'$ satisfying

$$\Sigma_{i\in I'} u_i \nabla q_i(x_{k+1}) + \Sigma_{j\in J'} v_j \nabla h_j(x_{k+1}) + \Sigma_{l\in L'} \xi_l \nabla p_l(x_{k+1}) = 0 \tag{19}$$

$$\Sigma_{i\in I'} u_i = 1 \tag{20}$$

$$(u_{I'}, v_{J'}, \xi_{L'}) \geq 0. \tag{21}$$

Notice that the set $I'$ is nonempty, due to the optimality of $(x_{k+1}, t_{k+1})$. It is now possible to relate the scalars $u_i, i \in I'$ and $v_j, j \in J'$ to an optimal solution of $(D_{\lambda_k})$.

LEMMA 9. *An optimal solution* $(\hat{y}, \hat{z})$ *of* $(D_{\lambda_k})$ *is given by*

$$\hat{y}_i = \begin{cases} 0 & \text{if } i \notin I' \\ u_i & \text{if } i \in I' \end{cases} \quad \hat{z}_j = \begin{cases} 0 & \text{if } j \notin J' \\ v_j & \text{if } j \in J' \end{cases}$$

*where* $u_i$ *and* $v_j$ *solve the system (19), (20), (21).*

*Proof.* From (20) and (21) it follows that $\hat{y}$ belongs to $\Sigma$ and $\hat{z} \geq 0$. Moreover, by the definition of $I'$ and $J'$ we obtain that

$$\Sigma_{i\in I'}\hat{y}_i q_i(x_{k+1}) + \Sigma_{j\in J'}\hat{z}_j h_j(x_{k+1}) = t_{k+1}.$$

Hence, since $(D_{\lambda_k})$ is the Lagrangian dual of $(\mathrm{P}_{\lambda_k})$, it follows using the previous equality, that

$$\begin{aligned}\Sigma_{i\in I'}\hat{y}_i q_i(x_{k+1}) + \Sigma_{j\in J'}\hat{z}_j h_j(x_{k+1}) &= \min_{x\in\mathcal{X}}\max_{i\in I}\{f_i(x) - \lambda_k g_i(x)\}\\ &= \max_{y\in\Sigma, z\geq 0} G(y, z, \lambda_k).\end{aligned}$$

It is left to show that $(\hat{y}, \hat{z})$ is an optimal solution of $\max_{y\in\Sigma,z\geq 0} G(y, z, \lambda_k)$. Since $\min_{x\in S}\{\hat{y}^\top q(x) + \hat{z}^\top h(x)\}$ is a convex optimization problem, the Karush–Kuhn–Tucker conditions are sufficient (Hiriart-Urruty and Lemaréchal, 1993). Clearly, by the definition of $(\hat{y}, \hat{z})$ and (19), (20), (21) the vector $x_{k+1}$ satisfies these conditions, and thus $x_{k+1}$ is an optimal solution of $\min_{x\in S}\{\hat{y}^\top q(x) + \hat{z}^\top h(x)\}$. Hence, $\hat{y} \in \Sigma, \hat{z} \geq 0$ satisfy

$$\begin{aligned}\max_{y\in\Sigma,z\geq 0} G(y, z, \lambda_k) &= \sum_{i\in I'}\hat{y}_i q_i(x_{k+1}) + \sum_{j\in J'}\hat{z}_j h_j(x_{k+1})\\ &= \min_{x\in\mathcal{S}}\{\hat{y}^\top q(x) + \hat{z}^\top h(x)\}\\ &= G(\hat{y}, \hat{z}, \lambda_k)\end{aligned}$$

and so $(\hat{y}, \hat{z})$ solves $(D_{\lambda_k})$.

This lemma provides an "easy" procedure to solve **Step** 1, and thus, to obtain the next iteration $(y_{k+1}, z_{k+1})$ of Algorithm 3. However, using this approach, a convex problem where the full set of constraints of $\mathcal{X}$ is present has to be solved in **Step** 1. Hence, the apparent advantage of this method over the dual method, whenever the feasible set $\mathcal{X}$ is formed by "easy" and "difficult" constraints, is here annulled.

Notice that the above result "extends" Lemma 3.4 in Barros *et al.* (1994). Moreover, we also know by Lemma 2 that $(y_{k+1}, z_{k+1})$ is an optimal solution of $(D_{\lambda_k})$ if and only if $y_{k+1}$ is an optimal solution of $(Q_{\lambda_k})$. At the same time, it follows that $z_{k+1}$ belongs to argmax $\{G(y_{k+1}, z, \lambda_{k+1}) : z \geq 0\}$. This observation and the next result permit to rank the next iteration value $\lambda_{k+1}$ of both methods.

LEMMA 10. *Assume that the Slater condition and either $(C_1)$ or $(C_2)$ holds then, $d(y, z) \leq c(y)$ for all $(y, z) \in \Sigma \times \mathcal{R}_+^r$. Moreover, for $(y_{k+1}, z_{k+1})$ an optimal solution of $(D_{\lambda_k})$ and $y_{k+1}$ an optimal solution of $(Q_{\lambda_k})$, $c(y_{k+1})$ equals $d(y_{k+1}, z_{k+1})$ if and only if $z_{k+1}$ belongs to*

$$\text{argmax}\{G(y_{k+1}, z, d(y_{k+1}, z_{k+1})) : z \geq 0\}.$$

*Proof.* Using the results of Dinkelbach (1967), we have

$$0 = \min_{x\in\mathcal{X}}\left\{y^\top(f(x) - c(y)g(x))\right\}$$

and

$$\begin{aligned} 0 &= \min_{x \in S} \left\{ y^\top (f(x) - d(y,z)g(x)) + z^\top h(x) \right\} \\ &\leq \min_{x \in \mathcal{X}} \left\{ y^\top (f(x) - d(y,z)g(x)) \right\}. \end{aligned}$$

Using again Dinkelbach (1967), the first result follows. For $(y_{k+1}, z_{k+1})$ being an optimal solution of $(D_{\lambda_k})$ it follows by Lagrangian duality that the above inequality is actually an equality if and only if $z_{k+1}$ belongs to

$$\operatorname{argmax}\{G(y_{k+1}, z, d(y_{k+1}, z_{k+1})) : z \geq 0\}.$$

Hence, we obtain that $d(y_{k+1}, z_{k+1}) = c(y_{k+1})$, and the result is proven.

The above lemma raises the question if in practice the situation $d(y_{k+1}, z_{k+1}) < c(y_{k+1})$ occurs frequently. According to our computational experience this situation does occur at the beginning of the application of the algorithms. This is to be expected in view of Lemma 10. Observe also that the computational effort required to compute $d(y_{k+1}, z_{k+1})$ can be expected to be less, when $S$ has "easy" constraints, than computing $c(y_{k+1})$.

### 3.2. Linear Case

We will now specialize the results derived in the previous section to the linear case. Hence, consider the generalized linear fractional programming problem defined by

$$\begin{aligned} f_i(x) &:= a_{i.}x + \alpha_i, g_i(x) := b_{i.}x + \beta_i \; \forall i \in I \text{ and} \\ \mathcal{X} &:= \{x \in \mathcal{R}^n : Cx \leq \gamma, x \geq 0\} \end{aligned}$$

where $a_{i.}$ and $b_{i.}$ denote respectively the $i$th row of the $m \times n$ matrix $A$ and $B$, $\alpha^\top = [\alpha_1, \ldots, \alpha_m]$, $\beta^\top = [\beta_1, \ldots, \beta_m]$ and $C$ a $q \times n$ matrix and $\gamma \in \mathcal{R}^q$. We will also assume:

$(A_1)$ *Feasibility assumption.* $\mathcal{X} \subseteq \mathcal{R}^n$ *is nonempty and bounded;*

$(A_2)$ *Positivity assumption.* $Bx + \beta > 0$ for all $x \in \mathcal{X}$.

Thus our generalized linear fractional programming problem is given by

$$\min_{x \in \mathcal{X}} \left\{ \max_{i \in I} \frac{a_{i.}x + \alpha_i}{b_{i.}x + \beta_i} \right\}. \tag{P}$$

Consider also the following optimization problem:

$$\min_{(x,x_0) \in \mathcal{X}_0} \left\{ \max_{i \in I} \frac{a_{i.}x + \alpha_i x_0}{b_{i.}x + \beta_i x_0} \right\} \tag{$P_0$}$$

with $\mathcal{X}_0 := \left\{(x, x_0) \in \mathcal{R}^{n+1}, Cx \leq \gamma x_0, \Sigma_{j=1}^n x_j + x_0 = 1, x \geq 0, x_0 \geq 0\right\}$. Before analyzing the relation between the above optimization problem and $(P)$, we will introduce the definition of equivalent problems (Craven, 1988).

DEFINITION 11. The two optimization problems

$$\max\{F(x) : x \in \mathcal{X}\} \text{and} \max\{G(x) : x \in \mathcal{Y}\}$$

are called equivalent if there exists a one-to-one mapping $\phi$ of the feasible set $\mathcal{X}$ onto $\mathcal{Y}$ such that $F(x) := G(\phi(x))$ for each $x \in \mathcal{X}$.

It is now possible to relate the two optimization problems $(P)$ and $(P_0)$.

LEMMA 12. *If* $(A_1)$ *and* $(A_2)$ *hold, then* $(P)$ *and* $(P_0)$ *are equivalent problems.*

*Proof.* In order to exhibit a one-to-one mapping of $\mathcal{X}$ onto $\mathcal{X}_0$, we will first show that for any $(x, x_0) \in \mathcal{X}_0$, the scalar $x_0$ can never be zero. Suppose that there exists some $(x, x_0) \in \mathcal{X}_0$ such that $x_0$ equals 0. Thus $Cx \leq 0$ and $\Sigma_{j=1}^n x_j = 1$ with $x_j \geq 0, j = 1, \ldots, n$. It follows now for any $w \in \mathcal{X}$ and $t > 0$ that

$$w + tx \geq 0 \text{ and } Cw + tCx \leq \gamma.$$

Hence, $w + tx \in \mathcal{X}$ for all $t > 0$ which contradicts the assumption $(A_1)$ that $\mathcal{X}$ is a bounded set. Consider the mapping $\phi$ of $\mathcal{X}$ into $\mathcal{X}_0$ given by $\phi(x) := \frac{1}{1+\Sigma_{j=1}^n x_j}(x, 1)$. Clearly, the image of $\mathcal{X}$ under $\phi$ is contained in $\mathcal{X}_0$. Moreover for all $(x, x_0) \in \mathcal{X}_0$ there exists a unique point in $\mathcal{X}$ given by $\frac{x}{x_0}$ and thus $\phi$ is a one-to-one mapping of $\mathcal{X}$ onto $\mathcal{X}_0$. Also, it follows easily that the objective value of $(P)$ at $x \in \mathcal{X}$ equals the objective value of $(P_0)$ at $\phi(x)$ which concludes the proof.

From the above lemma it follows that the denominators of $(P_0)$ are always positive for $(x, x_0) \in \mathcal{X}_0$. On the other hand, by assumption $(A_1)$ we obtain that $\mathcal{X}_0$ is a nonempty bounded set. Therefore, if assumptions $(A_1)$ and $(A_2)$ hold then the optimization problem $(P_0)$ corresponds to a standard generalized linear fractional problem.

Observe also that the mapping used in the proof of Lemma 12 is equivalent to the Charnes and Cooper transformation (Charnes and Cooper, 1962). Therefore, Lemma 12 is comparable to the results derived in Charnes and Cooper (1962) and in particular to the proof of equivalence between the different problems. However, while the Charnes and Cooper transformation is used to reduce a standard fractional linear programming problem into a linear programming problem, the transformation used in the context of Lemma 12 maintains the structure of the original problem.

The transformation of $(P)$ into $(P_0)$ will enable to apply directly the results derived for the nonlinear case in Section 3.1 to $(P_0)$. Indeed the feasible set of $(P_0)$ can be decomposed into the convex cone $\{(x, x_0) \in \mathcal{R}^{n+1} : C^\top x \leq \gamma x_0\}$ and the compact convex set $S_0 := \{(x, x_0) \in \mathcal{R}^{n+1} : x_0 + \Sigma_{j=1}^{n} x_j = 1, x \geq 0, x_0 \geq 0\}$. In order to derive the dual problem of $(P_0)$ the additional assumption $B^\top x + \beta x_0 > 0$ for all $(x, x_0) \in S_0$ is required. Observe this is guaranteed by the stronger positivity assumption ($A_2'$) $B > 0, \beta > 0$, used in Crouzeix *et al.* (1983, 1985); Jagannathan and Schaible (1983). We can now state the dual of $(P_0)$:

$$\max_{y \in \Sigma, z \geq 0} \inf_{(x,x_0) \in S_0} \frac{y^\top (Ax + \alpha x_0) + z^\top (Cx - \gamma x_0)}{y^\top (Bx + \beta x_0)} \tag{D$_0$}$$

formed by the constraints related to the original problem. The new algorithm described in the previous section constructs a sequence $(y_k, z_k) \in \Sigma \times \mathcal{R}_+^r$ with function values $d(y_k, z_k)$ approximating from below the optimal objective value of $(P_0)$. Remember that by Lemma 12 the value $\vartheta(P_0)$ equals $\vartheta(P)$. Hence, for a given $\lambda$ the new algorithm solves at **Step** 1 the parametric problem $(D_{0_\lambda})$:

$$\max_{y \in \Sigma, z \geq 0} \left\{ \min_{(x,x_0) \in S_0} \left\{ \left(y^\top (A - \lambda B) + z^\top C\right) x + \left(y^\top (\alpha - \lambda\beta) - z^\top \gamma\right) x_0 \right\} \right\}$$

The next iteration point, $(y, z)$, is given by an optimal solution of the above problem. It is left to evaluate the value of the objective function $d$ of $(D_0)$ at this point, i.e. computing $d(y, z)$. In this case, this corresponds to solving the following linear fractional programming problem:

$$\min_{(x,x_0) \in S_0} \frac{y^\top (Ax + \alpha x_0) + z^\top (Cx - \gamma x_0)}{y^\top (Bx + \beta x_0)} \tag{22}$$

Observe that the objective function in (22) is a ratio of linear functions, and thus quasiconcave. Since a quasiconcave function attains its minimum over a compact convex set at an extreme point (Avriel *et al.*, 1988) it follows that the optimal value of (22) has the following special form

$$d(y, z) = \min \left\{ \frac{\alpha^\top y - \gamma^\top z}{\beta^\top y}, \min_{1 \leq j \leq n} \frac{a_{.j}^\top y + c_{.j}^\top z}{b_{.j}^\top y} \right\} \tag{23}$$

where $a_{.j}$, $b_{.j}$ and $c_{.j}$ denote respectively the $j$th column of $A$, $B$ and $C$. This observation implies that ($D_0$) corresponds to the following generalized linear fractional programming problem:

$$\max_{y \in \Sigma, z \geq 0} \left\{ \min \left\{ \frac{\alpha^\top y - \gamma^\top z}{\beta^\top y}, \min_{1 \leq j \leq n} \frac{a_{.j}^\top y + c_{.j}^\top z}{b_{.j}^\top y} \right\} \right\} \tag{LD}$$

which is the standard dual problem of a generalized linear fractional program, described in Crouzeix *et al.* (1983, 1985); Jagannathan and Schaible (1983), under assumption ($A_2'$). Observe that the above dual problem can be derived using a weaker ($A_1$) assumption. In fact in Crouzeix *et al.* (1983, 1985); Jagannathan and

Schaible (1983) instead of $(A_1)$ it is only required that the feasible set should be non empty.

Crouzeix *et al.* (1985) discuss how to solve $(P)$ whenever the feasible set $\mathcal{X}$ is not bounded. In this case, they show that the Dinkelbach-type algorithm applied to the standard dual problem $(LD)$ converges and recuperates the optimal solution value. Therefore, it is appropriate to relate this approach to the new dual algorithm. Observe that the Dinkelbach-type algorithm applied to $(LD)$ requires solving the following parametric problem for a given $\lambda$:

$$\max_{y\in\Sigma, z\geq 0} \min\left\{(\alpha-\lambda\beta)^\top y-\gamma^\top z, \min_{1\leq j\leq n}\left\{(a_{.j}-\lambda b_{.j})^\top y+c_{.j}^\top z\right\}\right\}. \quad (\mathrm{LD}_\lambda)$$

However, due to the special form of (22) it follows that the above parametric problem corresponds to $(\mathrm{D0}_\lambda)$. Also, in the Dinkelbach-type algorithm the next iteration value is given by (23) and hence the two algorithms are identical. Therefore, the new dual algorithm introduced in Section 3.1 extends to the nonlinear case the Dinkelbach-type algorithm applied to the dual of a generalized linear fractional program, as suggested in Crouzeix *et al.* (1985). Nevertheless, it is important to stress in order to apply Algorithm 3 it is required that the feasible set $\mathcal{X}$ is compact.

Since for $(P)$ the corresponding set $S$ would be given by the noncompact set $\mathcal{R}^n_+$ while for $(P_0)$ the associated $S_0$ is compact, it follows that by considering $(P_0)$ instead of $(P)$ the results derived in the previous section can be smoothly applied to the linear case. Hence, both Lemmas 2 and 10 are valid and show that although the Dinkelbach-type algorithm applied to the standard dual and the dual algorithm of Barros *et al.* (1994) consider the same parametric function, the next iteration points taken by these two algorithms are different. Observe also that by specializing Proposition 7 we retrieve a sufficient condition to guarantee superlinear convergence for the Dinkelbach-type algorithm applied to a generalized linear fractional program. Indeed, due to the special form of (22), it follows that (22) has a unique solution if

$$\min\left\{\frac{\alpha^\top y_*-\gamma^\top z_*}{\beta^\top y_*}, \min_{1\leq j\leq n}\frac{a_{.j}^\top y_*+c_{.j}^\top z_*}{b_{.j}^\top y_*}\right\}$$

is uniquely attained. Therefore, the sufficient condition demands that for each optimal solution of $(LD)$ only one ratio is active. Observe this implies that at a neighborhood of the optimal point the associated parametric function is concave, see Proposition 4.1 of Crouzeix *et al.* (1985). Hence, in the neighborhood of the optimum, the Dinkelbach-type algorithm "coincides" with Newton's method, and thus its convergence rate is superlinear.

## 4. Scaled Algorithm

Following the same strategy used to derive the scaled version of the dual algorithm (Barros *et al.*, 1994) it is possible to construct the scaled version of the new algorithm introduced in the previous section. Before presenting this variant we introduce for $x_k \in S$ the vector-valued functions $f^{(k)}, g^{(k)}$ given by $f_i^{(k)}(x) := \frac{f_i(x)}{g_i(x_k)}$ and $g_i^{(k)}(x) := \frac{g_i(x)}{g_i(x_k)}$. Again, we will consider a convex generalized fractional programming problem and assume that the Slater condition holds.

We can consider the optimization problem

$$\max_{y\in\Sigma, z\geq 0} d^{(k)}(y, z) \tag{D$^{(k)}$}$$

with

$$d^{(k)}(y, z) := \min_{x\in S} \frac{y^\top f^{(k)}(x) + z^\top h(x)}{y^\top g^{(k)}(x)}$$

and its associated parametric problem given by

$$\max_{y\in\Sigma, z\geq 0} G^{(k)}(y, z, \lambda) \tag{D$_\lambda^{(k)}$}$$

with

$$G^{(k)}(y, z, \lambda) := \min_{x\in S} \left\{ y^\top \left( f^{(k)}(x) - \lambda g^{(k)}(x) \right) + z^\top h(x) \right\}.$$

Let now $(y_k, z_k)$ be an optimal solution of $(D_\lambda^{(k)})$ with $\lambda = d^{(k-1)}(y_{k-1}, z_{k-1})$, i.e.

$$(y_k, z_k) := \operatorname{argmax}_{y\in\Sigma, z\geq 0} G^{(k)}(y, z, \lambda).$$

In order to simplify the notation we will use, whenever there is no danger of confusion, $d'(y_k, z_k)$ instead of $d^{(k)}(y_k, z_k)$.

It is easy to show that $\vartheta(D^{(k)})$ equals $\vartheta(D)$. Moreover, it is also simple to establish an extension of Lemma 2 in terms of the scaled parametric functions. More precisely it follows that

$$\begin{aligned}
&G^{(k)}(y_k, z_k, d'(y_{k-1}, z_{k-1})) \\
&= \max_{y\in\Sigma, z\geq 0} \left\{ \min_{x\in S} \left\{ y^\top \left( f^{(k)}(x) - d'(y_{k-1}, z_{k-1}) g^{(k)}(x) \right) + z^\top h(x) \right\} \right\} \\
&= \max_{y\in\Sigma} \left\{ \min_{x\in\mathcal{X}} \left\{ y^\top \left( f^{(k)}(x) - d'(y_{k-1}, z_{k-1}) g^{(k)}(x) \right) \right\} \right\}.
\end{aligned}$$

Using the above equality and the convexity/concavity assumptions of the functions $f^{(k)}, g^{(k)}$ for all $k \geq 0$ on $S$ it follows now, by Von Neumann's min-max theorem that:

$$\begin{aligned}
&G^{(k)}(y_k, z_k, d'(y_{k-1}, z_{k-1})) \\
&= \min_{x\in\mathcal{X}} \left\{ \max_{y\in\Sigma} \left\{ y^\top \left( f^{(k)}(x) - d'(y_{k-1}, z_{k-1}) g^{(k)}(x) \right) \right\} \right\} \\
&= \min_{x\in\mathcal{X}} \left\{ \max_{i\in I} \left\{ f_i^{(k)}(x) - d'(y_{k-1}, z_{k-1}) g_i^{(k)}(x) \right\} \right\} \\
&= F^{(k)}(c'(y_{k-1}))
\end{aligned}$$

with $F^{(k)} : \mathcal{R} \longrightarrow \mathcal{R}$ the parametric function used in the Dinkelbach-type-2 algorithm (Crouzeix *et al.*, 1986). However, while in the Dinkelbach-type-2 algorithm $x_k$ is an optimal solution of the scaled parametric problem $(P_{\lambda_k}^{(k-1)})$, the vector $x_k$ in this variant must be an optimal solution of the fractional programming problem:

$$d'(y_{k-1}, z_{k-1}) := \min_{x \in S} \frac{y_{k-1}^\top f^{(k-1)}(x) + z_{k-1}^\top h(x)}{y_{k-1}^\top g^{(k-1)}(x)}$$

The scaled version of Algorithm 3 is described by the following procedure.

ALGORITHM 4.

**Step** $0'$.

If $g_i$ for all $i \in I$ are concave

Then Let $\lambda_0 := 0$ and take $x_0 \in \mathcal{X}$

Else Take $y_0 \in \Sigma$, $z_0 \geq 0$;

Compute $\lambda_0 := d'(y_0, z_0) = \min_{x \in S} \frac{y_0^\top f^{(0)}(x) + z_0^\top h(x)}{y_0^\top g^{(0)}(x)}$;

Let $x_1$ be an optimal solution of $d'(y_0, z_0)$;

Let $k := 1$;

**Step** $1'$.

Determine $(y_k, z_k) := \ \mathrm{argmax}_{y \in \Sigma, z \geq 0} G^{(k)}(y, z, \lambda_{k-1})$;

**Step** $2'$.

If $G^{(k)}(y_k, \lambda_{k-1}) = 0$

Then $(y_k, z_k)$ is an optimal solution of $(D^{(k)})$ with value $\lambda_{k-1}$ and Stop.

Else Goto **Step** $3'$;

**Step** $3'$.

Compute $\lambda_k := d'(y_k, z_k)$ and let $x_{k+1}$ be an optimal solution of $d'(y_k, z_k)$;

Let $k := k + 1$ and GoTo **Step** $1'$.

Similar to Barros *et al.* (1994) it can be shown that this scaled algorithm converges linearly, and that the sufficient condition of Proposition 7 also ensures that the rate of convergence of the scaled version becomes superlinear.

## 5. Computational Experience

In order to test the efficiency of the new dual algorithm, Algorithm 3 introduced in Section 3.1, we compared it with the Dinkelbach-type algorithm, Algorithm 2 and the dual algorithm, Algorithm 1. This comparison is also extended to the correspondent scaled versions of these three algorithms. Therefore, we used the same test problems as in Barros *et al.* (1994), i.e. we considered ratios with numerator quadratic functions $f_i(x) := \frac{1}{2} x^\top H_i x + a_i^\top x + b_i$, and denominator linear functions, $g_i(x) := c_i^\top x + d_i$. The quadratic functions, $f_i$, are generated in the following way:

- In the linear term each element of the vector $a_i$ is uniformly drawn from $[-15.0, 45.0]$. Similarly $b_i$ is drawn uniformly from $[-30.0, 0]$;
- The Hessian is defined by $H_i := L_i U_i L_i^\top$ where $L_i$ is a unit lower triangular matrix with components uniformly drawn from $[-2.5, 2.5]$ and $U_i$ is a positive diagonal matrix, with elements uniformly drawn from [0.1,1.6]. When a positive semidefinite Hessian is required the first component of the diagonal matrix is set to zero.

The linear functions, $g_i$, are constructed using a similar procedure: each element of the vector $c_i$ is uniformly drawn from [0.0,10.0]. Similarly $d_i$ is drawn uniformly from [1.0, 5.0]. Finally, the feasible domains considered are the following:

$$\mathcal{X}_1 := \{x \in S : \Sigma_{j=1}^n x_j \leq 1\}$$
$$\mathcal{X}_2 := \{x \in S : \Sigma_{j \in J_1} x_j \leq 1, \Sigma_{j \in J_2} x_j \leq 1\}$$

with $S := \{x \in \mathcal{R}^n : 0 \leq x_j \leq 1, j = 1, \ldots, n\}$ and $J_1 := \{1 \leq j \leq n : j \text{ is odd}\}$ and $J_2 := \{1 \leq j \leq n : j \text{ is even}\}$.

Both methods were implemented in Sun Pascal, linked to a pair of existing routines written in Sun FORTRAN and ran on a Sun Sparc System 600 workstation, using the default double precision (64-bit IEEE floating point format) real numbers of Sun Pascal and FORTRAN. Both compilers were used with the default compilation options.

For the minimization of the maximum of quadratic functions with linear constraints we used the bundle trust method coded in FORTRAN (Outrata *et al.*, 1991). In the dual type algorithms **Steps** 1 and $1'$ are solved by computing the correspondent minimal ellipsoidal norm problem, see Barros (1995). The fractional programming problem that occurs in **Steps** 0, 3, $0'$ and $3'$ of the dual type algorithms is solved by the classical Dinkelbach algorithm (Dinkelbach, 1967). The code used to solve the above quadratic problems is an implementation in FORTRAN of Lemke's algorithm (Ravindran, 1972).

In Algorithm 1 we used in **Step** 0 the initial point $y_0^\top := (\frac{1}{m}, \ldots, \frac{1}{m})$, while in the new dual algorithm the initial points in **Step** 0 are given by $y_0^\top := (\frac{1}{m}, \ldots, \frac{1}{m})$ and $z_0^\top := (0, \ldots, 0)$. In the Dinkelbach-type algorithm we take in **Step** 0:

$$\lambda_1 := d(y_0, z_0) = \min_{x \in S} \frac{y_0^\top f(x)}{y_0^\top g(x)}.$$

The tolerance used in both implementations is $\epsilon := 5 \times 10^{-6}$, see Barros (1995); Barros *et al.* (1994).

The results of the computational experience are summarized in the following tables. For each pair $(n, m)$, where $n$ is the number of variables and $m$ the number of ratios, 5 uncorrelated instances of the problem were generated and solved by these algorithms. Hence, the entries of the tables are averages of the corresponding values. The columns under *Dinkel* report the results obtained using the Dinkelbach-type algorithm for several ratios. Similarly, the columns under *Dual* report the results obtained using Algorithm 1, while *NDual* report the results obtained using the new dual algorithm presented in Section 3.1. In these cases two extra columns are presented concerning important steps of these algorithms. Hence, column $\%Fr$ refers to the percentage of the time used to compute the next iteration point, i.e. $c(y_k)$, respectively $d(y_k, z_k)$, while Column $\%K$ refers to the percentage of the time used to solve the Karush–Kuhn–Tucker system and thus obtaining $y_{k+1}$, respectively $(y_{k+1}, z_{k+1})$. The column *It* refers to the number of iterations performed by the corresponding algorithm. Each *Sec* column refers to the execution time in seconds of the mentioned Sun workstation measured by the available standard clock function of the Sun Pascal library. This measures the elapsed execution time from the start to the end of the corresponding method, excluding input and output operations. Finally under the column $\%Imp.$ we report the percentage of improvement in total execution time between the three different algorithms tested. Thus, the percentage of improvement in total execution time of the dual type algorithms over the Dinkelbach-type algorithm, are contained in column $DiD$, i.e. $(1 - \frac{Sec(Dual)}{Sec(Din)} \times 100$ and column *DiND*, i.e. $(1 - \frac{Sec(Dual)}{Sec(Din)}) \times 100$. Finally, column *NDD* contains the percentage of improvement in total execution time of the dual algorithm over the new dual algorithm, i.e. $(1 - \frac{Sec(Dual)}{Sec(N\ Dual)}) \times 100$.

Tables I and II contain the results obtained for test problems where the quadratic functions, $f_i$, are strictly convex. In these cases the convergence rate of both dual algorithms is superlinear, see Barros *et al.* (1994) and Proposition 7. From these results it seems that the new dual is better in terms of number of iterations than the Dinkelbach-type algorithm. However, this improvement is not as effective in terms of execution time, in particular, for the test problems with feasible set $\mathcal{X}_1$. Observe, on average more iterations are required by the new dual algorithm than Algorithm 1. Furthermore, the Algorithm 1 has a much better performance than the new dual.

Tables III and IV contain the results obtained for test problems where the quadratic functions, $f_i$, are only convex. The results resumed in these two tables show that the behavior of the new dual algorithm worsens in the case where the functions $f_i$ are no longer strictly convex. Indeed, both in terms of number of iterations and execution time the performance of the new dual algorithm is not so often better than the one of the Dinkelbach-type algorithm. Again, for the test problems with feasible region $\mathcal{X}_2$ the new dual algorithm has a slightly better performance. However, the dual algorithm (Barros *et al.*, 1994) still has a better performance than the new dual algorithm, and the Dinkelbach-type algorithm.

Tables V and VI contain the computational results obtained with the scaled version of the mentioned algorithms, and using $x_0^\top := (0, \ldots, 0)$. In these tables the columns under *Dinkel-2* report the results obtained using the Dinkelbach-type-2 algorithm (Crouzeix *et al.*, 1986). Similarly, the columns under *Dual-2* report the results obtained using the scaled version of Algorithm 1 (Barros *et al.*, 1994), while *NDual-2* report the results obtained using the scaled version of the new dual

TABLE I. $\mathcal{X}_1$ and strictly quasiconvex ratios

| Prob. | | Dinkel | | Dual | | | | NDual | | | | % Imp. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $m$ | $It$ | $Sec$ | $It$ | $\%Fr$ | $\%K$ | $Sec$ | $It$ | $\%Fr$ | $\%K$ | $Sec$ | $DiD$ | $DiND$ | $NDD$ |
| 5 | 5 | 8 | 0.87 | 3 | 16.3 | 1.6 | 0.64 | 6 | 12.7 | 4.5 | 1.49 | 26.4 | −71.3 | 57.1 |
| 10 | 5 | 11 | 10.45 | 3 | 8.1 | 1.4 | 4.08 | 6 | 8.0 | 1.2 | 7.43 | 61.0 | 28.8 | 45.2 |
| 15 | 5 | 9 | 16.92 | 3 | 18.4 | 1.8 | 7.57 | 6 | 13.5 | 2.4 | 12.62 | 55.2 | 25.4 | 40.0 |
| 20 | 5 | 8 | 33.81 | 3 | 9.0 | 0.7 | 21.33 | 7 | 8.4 | 0.9 | 42.14 | 36.9 | −24.6 | 49.4 |
| 5 | 10 | 9 | 1.51 | 4 | 15.5 | 3.8 | 0.53 | 6 | 12.4 | 4.3 | 0.98 | 64.6 | 34.8 | 45.7 |
| 10 | 10 | 14 | 12.22 | 4 | 10.5 | 1.1 | 4.57 | 7 | 8.7 | 1.5 | 8.22 | 62.6 | 32.7 | 44.4 |
| 15 | 10 | 9 | 18.29 | 3 | 10.4 | 1.1 | 11.51 | 7 | 10.5 | 1.8 | 19.26 | 37.1 | −5.3 | 40.2 |
| 20 | 10 | 10 | 53.07 | 3 | 10.0 | 0.8 | 25.32 | 7 | 8.7 | 0.9 | 46.26 | 52.3 | 12.8 | 45.3 |
| 5 | 15 | 8 | 3.02 | 3 | 8.9 | 3.1 | 1.01 | 8 | 6.5 | 2.9 | 2.50 | 66.7 | 17.4 | 59.7 |
| 10 | 15 | 11 | 11.39 | 3 | 10.5 | 1.2 | 4.76 | 8 | 9.3 | 1.1 | 9.81 | 58.2 | 13.9 | 51.5 |
| 15 | 15 | 9 | 26.59 | 3 | 10.3 | 1.0 | 14.06 | 8 | 8.7 | 1.1 | 27.47 | 47.1 | −3.3 | 48.8 |
| 20 | 15 | 12 | 71.10 | 3 | 9.5 | 0.8 | 28.96 | 8 | 9.2 | 1.0 | 59.86 | 59.3 | 15.8 | 51.6 |
| 5 | 20 | 9 | 1.58 | 4 | 10.7 | 2.4 | 0.99 | 8 | 10.1 | 4.2 | 1.91 | 37.3 | −21.1 | 48.2 |
| 10 | 20 | 11 | 13.95 | 4 | 10.5 | 1.6 | 5.50 | 8 | 8.7 | 1.7 | 10.70 | 60.6 | 23.3 | 48.6 |
| 15 | 20 | 11 | 34.01 | 4 | 9.3 | 0.9 | 14.90 | 7 | 8.6 | 1.0 | 27.52 | 56.2 | 19.1 | 45.9 |
| 20 | 20 | 13 | 77.23 | 3 | 9.6 | 0.8 | 34.87 | 7 | 8.9 | 1.0 | 64.66 | 54.9 | 16.3 | 46.1 |

TABLE II. $\mathcal{X}_2$ and strictly quasiconvex ratios

| Prob. | | Dinkelb | | Dual | | | | NDual | | | | %Imp. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $m$ | $It$ | $Sec$ | $It$ | $\%Fr$ | $\%K$ | $Sec$ | $It$ | $\%Fr$ | $\%K$ | $Sec$ | $DiD$ | $DiND$ | $NDD$ |
| 5 | 5 | 8 | 2.15 | 2 | 10.2 | 0.4 | 0.79 | 4 | 7.8 | 1.3 | 1.30 | 63.1 | 39.7 | 38.8 |
| 10 | 5 | 11 | 13.08 | 3 | 9.5 | 0.8 | 5.65 | 6 | 7.6 | 0.9 | 9.04 | 56.8 | 30.9 | 37.6 |
| 15 | 5 | 9 | 23.66 | 3 | 11.4 | 0.8 | 11.51 | 5 | 10.0 | 1.1 | 14.64 | 51.4 | 38.1 | 21.4 |
| 20 | 5 | 9 | 43.54 | 3 | 9.9 | 0.6 | 21.89 | 5 | 9.0 | 0.8 | 39.16 | 49.7 | 10.1 | 44.1 |
| 5 | 10 | 12 | 2.43 | 4 | 12.9 | 2.2 | 0.88 | 6 | 10.2 | 2.5 | 1.09 | 63.7 | 55.2 | 18.9 |
| 10 | 10 | 13 | 12.33 | 3 | 8.8 | 0.7 | 5.47 | 6 | 8.4 | 0.8 | 8.62 | 55.6 | 30.1 | 36.5 |
| 15 | 10 | 13 | 38.49 | 3 | 8.6 | 0.6 | 16.74 | 6 | 8.4 | 0.8 | 26.53 | 56.5 | 31.1 | 36.9 |
| 20 | 10 | 10 | 58.73 | 3 | 9.8 | 0.6 | 31.17 | 5 | 8.0 | 0.7 | 47.01 | 46.9 | 20.0 | 33.7 |
| 5 | 15 | 9 | 1.80 | 3 | 15.8 | 3.1 | 0.77 | 5 | 15.6 | 3.3 | 1.17 | 57.0 | 35.1 | 33.7 |
| 10 | 15 | 11 | 14.79 | 3 | 8.0 | 0.7 | 6.97 | 7 | 7.4 | 0.8 | 12.45 | 52.9 | 15.8 | 44.0 |
| 15 | 15 | 9 | 30.97 | 3 | 8.9 | 0.7 | 16.92 | 6 | 7.9 | 0.7 | 29.13 | 45.4 | 6.0 | 41.9 |
| 20 | 15 | 11 | 75.05 | 3 | 9.1 | 0.6 | 35.01 | 6 | 8.7 | 0.7 | 62.91 | 53.3 | 16.2 | 44.3 |
| 5 | 20 | 8 | 1.99 | 4 | 13.0 | 2.0 | 1.13 | 7 | 7.2 | 2.1 | 1.97 | 43.0 | 1.2 | 42.4 |
| 10 | 20 | 11 | 15.10 | 3 | 8.2 | 0.9 | 7.68 | 6 | 8.0 | 1.2 | 12.02 | 49.2 | 20.4 | 36.1 |
| 15 | 20 | 13 | 38.51 | 3 | 9.1 | 0.9 | 16.52 | 6 | 8.0 | 0.9 | 31.09 | 57.1 | 19.3 | 46.9 |
| 20 | 20 | 11 | 88.73 | 3 | 8.0 | 0.5 | 38.61 | 7 | 7.3 | 0.6 | 78.78 | 56.5 | 11.2 | 51.0 |

TABLE III. $\mathcal{X}_1$ and semistrictly quasiconvex ratios

| Prob. | | Dinkel | | Dual | | | | NDual | | | | %Imp. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $m$ | $It$ | $Sec$ | $It$ | $\%Fr$ | $\%K$ | $Sec$ | $It$ | $\%Fr$ | $\%K$ | $Sec$ | $DiD$ | $DiND$ | $NDD$ |
| 5 | 5 | 8 | 0.60 | 5 | 13.1 | 7.6 | 0.47 | 12 | 10.6 | 6.8 | 1.20 | 21.7 | −100.6 | 60.9 |
| 10 | 5 | 14 | 7.32 | 3 | 10.7 | 2.2 | 2.15 | 8 | 12.0 | 2.3 | 4.54 | 70.6 | 38.0 | 52.6 |
| 15 | 5 | 11 | 19.25 | 3 | 10.0 | 1.4 | 8.85 | 10 | 10.3 | 1.9 | 22.84 | 54.0 | −18.7 | 61.2 |
| 20 | 5 | 11 | 36.75 | 3 | 9.5 | 1.1 | 17.52 | 7 | 8.9 | 1.4 | 33.92 | 52.3 | 7.7 | 48.3 |
| 5 | 10 | 12 | 2.14 | 4 | 8.3 | 5.6 | 0.99 | 14 | 10.6 | 4.9 | 3.25 | 53.6 | −51.7 | 69.4 |
| 10 | 10 | 10 | 13.91 | 4 | 6.6 | 1.4 | 5.73 | 12 | 6.6 | 1.4 | 15.37 | 58.8 | −10.5 | 62.7 |
| 15 | 10 | 10 | 20.07 | 3 | 7.2 | 1.5 | 8.30 | 7 | 10.3 | 2.0 | 18.79 | 58.6 | 6.4 | 55.8 |
| 20 | 10 | 12 | 66.09 | 3 | 7.9 | 1.0 | 27.52 | 11 | 7.8 | 1.2 | 70.62 | 58.4 | −6.9 | 61.0 |
| 5 | 15 | 7 | 2.82 | 4 | 5.3 | 1.1 | 2.22 | 11 | 4.0 | 2.3 | 4.37 | 21.2 | −55.1 | 49.2 |
| 10 | 15 | 11 | 14.45 | 3 | 7.6 | 1.2 | 4.46 | 10 | 8.7 | 1.3 | 11.88 | 69.1 | 17.8 | 62.4 |
| 15 | 15 | 11 | 36.63 | 3 | 6.8 | 1.0 | 17.62 | 11 | 7.0 | 1.2 | 43.93 | 51.9 | −19.9 | 59.9 |
| 20 | 15 | 11 | 62.41 | 3 | 8.2 | 0.9 | 29.53 | 9 | 8.6 | 1.1 | 68.14 | 52.7 | −9.2 | 56.7 |
| 5 | 20 | 11 | 2.33 | 4 | 11.6 | 2.4 | 0.91 | 13 | 12.2 | 5.1 | 2.57 | 60.9 | −10.3 | 64.6 |
| 10 | 20 | 11 | 14.89 | 4 | 9.7 | 2.3 | 6.04 | 10 | 8.8 | 1.6 | 14.46 | 59.4 | 2.9 | 58.2 |
| 15 | 20 | 12 | 33.14 | 3 | 8.0 | 1.2 | 13.09 | 9 | 8.7 | 1.4 | 31.70 | 60.5 | 4.4 | 58.7 |
| 20 | 20 | 12 | 84.98 | 4 | 6.1 | 0.7 | 38.26 | 12 | 7.0 | 0.8 | 95.54 | 55.0 | −12.4 | 60.0 |

TABLE IV. $\mathcal{X}_2$ and semistrictly quasiconvex ratios

| Prob. | | Dinkel | | Dual | | | | NDual | | | | %Imp. | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $m$ | $It$ | $Sec$ | $It$ | $\%Fr$ | $\%K$ | $Sec$ | $It$ | $\%Fr$ | $\%K$ | $Sec$ | $DiD$ | $DiND$ | $NDD$ |
| 5 | 5 | 8 | 0.49 | 7 | 22.5 | 3.0 | 0.62 | 9 | 17.7 | 11.7 | 0.82 | −26.4 | −66.3 | 24.0 |
| 10 | 5 | 13 | 8.53 | 3 | 15.7 | 2.9 | 2.38 | 7 | 12.9 | 2.7 | 5.02 | 72.1 | 41.2 | 52.6 |
| 15 | 5 | 9 | 35.09 | 3 | 7.3 | 0.6 | 20.22 | 9 | 6.0 | 0.8 | 43.04 | 42.4 | −22.7 | 53.0 |
| 20 | 5 | 11 | 47.46 | 3 | 8.9 | 0.8 | 25.84 | 6 | 8.2 | 0.9 | 41.35 | 45.6 | 12.9 | 37.5 |
| 5 | 10 | 13 | 2.91 | 5 | 16.0 | 1.6 | 1.08 | 11 | 10.5 | 3.1 | 2.18 | 63.0 | 25.1 | 50.6 |
| 10 | 10 | 9 | 13.32 | 3 | 11.9 | 0.9 | 3.92 | 9 | 10.0 | 1.8 | 10.33 | 70.6 | 22.4 | 62.1 |
| 15 | 10 | 11 | 24.04 | 3 | 9.0 | 1.3 | 12.03 | 8 | 9.6 | 1.6 | 21.77 | 50.0 | 9.4 | 44.8 |
| 20 | 10 | 10 | 69.11 | 3 | 7.7 | 0.5 | 38.73 | 7 | 7.4 | 0.7 | 61.41 | 44.0 | 11.1 | 36.9 |
| 5 | 15 | 9 | 4.36 | 4 | 15.0 | 2.9 | 1.15 | 8 | 9.2 | 2.1 | 2.48 | 73.6 | 43.1 | 53.5 |
| 10 | 15 | 10 | 13.75 | 3 | 9.4 | 0.9 | 5.65 | 9 | 8.6 | 1.0 | 12.21 | 58.9 | 11.2 | 53.7 |
| 15 | 15 | 12 | 51.96 | 3 | 6.8 | 0.5 | 21.59 | 8 | 6.4 | 0.7 | 42.82 | 58.5 | 17.6 | 49.6 |
| 20 | 15 | 10 | 70.29 | 3 | 8.5 | 0.6 | 35.50 | 8 | 7.3 | 0.7 | 75.72 | 49.5 | −7.7 | 53.1 |
| 5 | 20 | 14 | 4.09 | 4 | 13.6 | 3.0 | 1.15 | 9 | 13.7 | 4.2 | 1.65 | 71.9 | 59.5 | 30.4 |
| 10 | 20 | 10 | 16.72 | 3 | 8.0 | 1.0 | 6.78 | 10 | 7.0 | 1.1 | 17.18 | 59.5 | −2.7 | 60.5 |
| 15 | 20 | 11 | 42.89 | 3 | 6.8 | 0.5 | 19.98 | 9 | 6.3 | 0.7 | 47.52 | 53.4 | −10.8 | 58.0 |
| 20 | 20 | 12 | 104.54 | 4 | 7.3 | 0.4 | 45.37 | 12 | 6.2 | 0.6 | 113.72 | 56.6 | −8.8 | 60.1 |

TABLE V. Strictly quasiconvex ratios

| | | $\mathcal{X}_1$ | | | | | | $\mathcal{X}_2$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Prob. | | Dink-2 | | Dual-2 | | NDual-2 | | Dink-2 | | Dual-2 | | NDual-2 | |
| $n$ | $m$ | $It$ | $Sec$ | $It$ | $Sec$ | $It$ | $Sec$ | $It$ | $Sec$ | $It$ | $Sec$ | $It$ | $Sec$ |
| 5 | 5 | 5 | 0.75 | 3 | 0.68 | 6 | 1.66 | 6 | 1.81 | 3 | 1.11 | 5 | 1.84 |
| 10 | 5 | 6 | 7.79 | 3 | 4.71 | 7 | 9.52 | 6 | 11.98 | 3 | 6.37 | 6 | 11.00 |
| 15 | 5 | 6 | 15.49 | 3 | 8.73 | 6 | 15.19 | 6 | 19.01 | 3 | 11.81 | 5 | 16.34 |
| 20 | 5 | 6 | 37.54 | 3 | 20.71 | 7 | 44.71 | 6 | 32.55 | 3 | 20.65 | 5 | 37.40 |
| 5 | 10 | 5 | 0.62 | 3 | 1.46 | 6 | 2.01 | 6 | 1.67 | 3 | 0.77 | 5 | 1.46 |
| 10 | 10 | 6 | 6.60 | 3 | 4.34 | 7 | 9.90 | 6 | 7.45 | 3 | 5.78 | 6 | 8.44 |
| 15 | 10 | 6 | 16.27 | 3 | 12.39 | 7 | 23.97 | 6 | 21.77 | 3 | 17.26 | 6 | 26.54 |
| 20 | 10 | 6 | 43.85 | 3 | 31.47 | 7 | 72.09 | 5 | 47.10 | 3 | 31.13 | 6 | 52.31 |
| 5 | 15 | 6 | 2.74 | 3 | 2.03 | 8 | 3.14 | 6 | 1.22 | 3 | 1.75 | 6 | 1.74 |
| 10 | 15 | 6 | 8.46 | 3 | 6.64 | 8 | 11.44 | 6 | 10.68 | 3 | 7.49 | 7 | 13.76 |
| 15 | 15 | 6 | 22.41 | 3 | 13.72 | 8 | 31.93 | 7 | 29.20 | 3 | 16.84 | 6 | 28.44 |
| 20 | 15 | 6 | 51.63 | 3 | 34.03 | 8 | 72.36 | 6 | 51.93 | 3 | 34.17 | 6 | 66.23 |
| 5 | 20 | 5 | 1.46 | 4 | 1.02 | 8 | 2.57 | 5 | 1.52 | 3 | 1.30 | 7 | 2.22 |
| 10 | 20 | 6 | 8.13 | 4 | 8.11 | 8 | 12.20 | 6 | 9.33 | 3 | 10.45 | 6 | 15.17 |
| 15 | 20 | 6 | 25.35 | 4 | 17.57 | 8 | 30.93 | 6 | 24.86 | 3 | 15.23 | 7 | 29.66 |
| 20 | 20 | 6 | 56.72 | 3 | 31.36 | 8 | 63.29 | 6 | 60.00 | 3 | 39.96 | 7 | 82.28 |

TABLE VI. Semistrictly quasiconvex ratios

| | | $\mathcal{X}_1$ | | | | | | $\mathcal{X}_2$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Prob. | | Dink-2 | | Dual-2 | | NDual-2 | | Dink-2 | | Dual-2 | | NDual-2 | |
| $n$ | $m$ | $It$ | $Sec$ | $It$ | $Sec$ | $It$ | $Sec$ | $It$ | $Sec$ | $It$ | $Sec$ | $It$ | $Sec$ |
| 5 | 5 | 5 | 0.53 | 6 | 0.84 | 12 | 1.13 | 5 | 0.37 | 7 | 0.84 | 10 | 1.31 |
| 10 | 5 | 6 | 3.83 | 3 | 2.45 | 8 | 5.58 | 7 | 5.24 | 3 | 3.79 | 7 | 6.03 |
| 15 | 5 | 6 | 13.72 | 3 | 10.48 | 10 | 32.21 | 6 | 28.32 | 4 | 23.07 | 10 | 50.78 |
| 20 | 5 | 5 | 20.76 | 3 | 14.09 | 7 | 30.01 | 6 | 33.00 | 3 | 23.14 | 7 | 40.14 |
| 5 | 10 | 6 | 1.53 | 4 | 1.94 | 14 | 5.30 | 7 | 1.97 | 5 | 1.25 | 11 | 3.23 |
| 10 | 10 | 6 | 12.41 | 3 | 9.66 | 12 | 20.81 | 6 | 11.63 | 3 | 5.85 | 8 | 11.52 |
| 15 | 10 | 5 | 15.59 | 3 | 10.38 | 7 | 34.34 | 6 | 18.66 | 3 | 13.77 | 8 | 23.61 |
| 20 | 10 | 6 | 39.35 | 3 | 30.72 | 11 | 70.90 | 6 | 45.86 | 3 | 37.44 | 7 | 60.98 |
| 5 | 15 | 6 | 3.94 | 4 | 2.40 | 11 | 4.08 | 6 | 3.56 | 4 | 1.67 | 8 | 2.63 |
| 10 | 15 | 6 | 7.95 | 3 | 5.28 | 10 | 12.36 | 6 | 10.50 | 3 | 6.22 | 9 | 13.51 |
| 15 | 15 | 6 | 25.73 | 4 | 20.06 | 11 | 49.00 | 6 | 33.27 | 3 | 21.20 | 8 | 44.73 |
| 20 | 15 | 6 | 52.98 | 3 | 31.32 | 10 | 84.40 | 7 | 60.97 | 3 | 31.31 | 9 | 70.72 |
| 5 | 20 | 6 | 1.37 | 4 | 1.97 | 12 | 3.11 | 6 | 1.56 | 4 | 1.48 | 8 | 1.87 |
| 10 | 20 | 6 | 12.81 | 4 | 6.96 | 10 | 21.42 | 6 | 13.05 | 3 | 9.04 | 11 | 22.80 |
| 15 | 20 | 6 | 21.24 | 3 | 15.09 | 9 | 32.31 | 6 | 38.13 | 3 | 21.94 | 9 | 49.75 |
| 20 | 20 | 6 | 54.79 | 4 | 39.71 | 12 | 98.10 | 6 | 69.49 | 3 | 50.33 | 12 | 116.22 |

algorithm presented in Section 4. Since the total computational time used by the scaled versions of the dual-type algorithms appeared to be distributed in a similar way as for its original version these tables are presented in a more condensed form. Tables V and VI show that the Dinkelbach-type-2 algorithm dominates, both in iteration number and execution time, the Dinkelbach-type algorithm. On the other hand, the scaling of both the dual-type algorithms does not appear to produce significant improvements on the behavior of the original algorithms. Contrary to the dual algorithm (Barros *et al.*, 1994) the new dual algorithm and its scaled version no longer dominate the Dinkelbach-type-2 algorithm.

## 6. Conclusions

The usual approach to generalized fractional programming is usually a primal approach due to the "awkward" form of the standard dual problem of a generalized fractional program. Recently, Barros *et al.* (1994) proposed a dual algorithm for generalized fractional programrning by means of an alternative dual. However, it was left to investigate if the standard dual could actually be solved efficiently. This paper answers this question by introducing a new algorithm which solves in an efficient way this "awkward" dual. Moreover, this algorithm extends to the nonlinear case the Dinkelbach-type algorithm applied to the standard dual of a generalized linear fractional program. Therefore, it can be seen as an extension of a Dinkelbach-type algorithm to the nonlinear case with a "difficult" parametric problem. However, under some reasonable assumptions it is possible to solve efficiently this parametric problem in the nonlinear case. Moreover, due to information provided by the dual problem it is possible to derive better rate of convergence results for the new algorithm than for the Dinkelbach-type algorithm applied to the primal problem. Finally, the approach developed in this paper also permits to show that the standard duality results for the special case of generalized linear fractional programs with a compact feasible region can be easily derived by specializing the duality results for the more general nonlinear case.

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## References

Avriel, M., Diewert, W., Schaible, S., and Zang, I. (1988). *Generalized Concavity*, volume 36 of *Mathematical Concepts and Methods in Science and Engineering*. Plenum Press, New York.

Barros, A. (1995). *Discrete and Fractional Programming Techniques for Location Models*, volume 89 of *Tinbergen Institute Research Series*. Thesis Publishers, Amsterdam.

Barros, A., Frenk, J., Schaible, S., and Zhang, S. (1994). A new algorithm for generalized fractional programs. Technical Report TI-94-23, Tinbergen Institute Rotterdam. To appear in *Mathematical Programming*.

Benadada, Y. (1989). *Approches de résolution du probléme de programmation fractionnaire généralisée*. Ph.D. thesis, Département d'informatique et de recherche opérationnelle, Université de Montréal, Montréal, Canada.

Bernard, J. and Ferland, J. (1989). Convergence of interval-type algorithms for generalized fractional programming. *Mathematical Programming*, **43**, 349–364.

Bohnenblust, H., Karlin, S., and Shapley, L. (1950). Solutions of discrete two-person games. *Annals of Mathematics Studies*, **24**, 51–72.

Charnes, A. and Cooper, W. (1962). Programming with linear fractional functionals. *Naval Research Logistics Quarterly*, **9**, 181–186.

Craven, B. (1988). *Fractional Programming*. Heldermann-Verlag, Berlin.

Crouzeix, J. and Ferland, J. (1991). Algorithms for generalized fractional programming. *Mathematical Programming*, **52**(2), 191–207.

Crouzeix, J., Ferland, J., and Schaible, S. (1983). Duality in generalized linear fractional programming. *Mathematical Programming*, **27**(3), 342–354.

Crouzeix., J., Ferland, J., and Schaible, S. (1985). An algorithm for generalized fractional programs. *Journal of Optimization Theory and Applications*, **47**(1), 35–49.

Crouzeix, J., Ferland, J., and Schaible, S. (1986). A note on an algorithm for generalized fractional programs. *Journal of Optimization Theory and Applications*, **50**(1), 183–187.

Dinkelbach, W. (1967). On nonlinear fractional programming. *Management Science*, **13**(7), 492–498.

Ferland, J. and Potvin, J. (1985). Generalized fractional programming: Algorithms and numerical experimentation. *European Journal of Operational Research*, **20**, 92–101.

Hiriart-Urruty, J. and Lemaréchal, C. (1993). *Convex Analysis and Minimization Algorithms I: Fundamentals*, volume 1. Springer-Verlag, Berlin.

Jagannathan, R. and Schaible, S. (1983). Duality in generalized fractional programming via Farkas' lemma. *Journal of Optimization Theory and Applications*, **41**(3), 417–424.

Outrata, J., Schramm, H., and Zowe, J. (1991). Bundle trust methods: Fortran codes for nondifferentiable optimization. User's guide. Technical Report 269, Mathematisches Institut. Universität Bayreuth.

Pardalos, P.M. and Phillips, A.T. (1991). Global optimization of fractional programs. *Journal of Global Optimization*, **1**(2), 173–182.

Pshenichnyi, B. (1971). *Necessary Conditions for an Extremum*. Marcel Dekker Inc., New York.

Ravindran, A. (1972). A computer routine for quadratic and linear programming problems. *Communications of the ACM*, **15**(9), 818.

Rockafellar, R. (1970). *Convex Analysis*. Princeton University Press, Princeton, New Jersey.

Rockafellar, R. (1983). Generalized subgradients in mathematical programminng. In A. Bachem, M. Grötschel, and B. Korte, editors, *Mathematical Programming. The State of the Art*, chapter 2, pages 368–390. Springer-Verlag.

Schaible, S. (1978). *Analyse und Anwendungen von Quotientenprogrammen, ein Beitrag zur Planung mit Hilfe der Nichtlinearen Programmierung*, volume 42 of *Mathematical Systems in Economics*. Hain-Verlag, Meisenheim.

Schaible, S. (1983). Fractional programming. *Zeitschrift für Operations Research*, **27**, 39–54.

Schaible, S. and Ibaraki, T. (1983). Fractional programming (Invited Review). *European Journal of Operational Research*, **12**, 325–338.

Sion, M. (1958). On general minimax theorems. *Pacific Journal of Mathematics*, **8**, 171–176.

Werner, J. (1988). Duality in generalized fractional programming. In K. Hoffman, J. Hiriart-Urruty, C. Lemaréchal, and J. Zowe, editors, *Trends in Mathematical Optimization*, International Series of Numerical Mathematics, pages 197–232. Birkhäuser-Verlag Basel.